Logic as Based on Incompatibility

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Abstract

The aim of the paper is to tackle two related questions: *Is it possible to reduce the foundations of logic to the mere concept of incompatibility?* and *Does this reduction lead us to a specific logical system?* We conclude that the answers, respectively, are YES and a qualified NO (qualified in the sense that basing semantics on incompatibility does make some logical systems more natural than others, but without ruling out the alternatives.)

1 Can inference serve as a foundation of logic?

Can we base the whole of logic solely on the concept of incompatibility? My motivation for asking this is two-fold: firstly, a technical interest in what a minimal foundations of logic might be; and secondly, the existence of philosophers who have taken incompatibility as the ultimate key to human reason (*viz.*, e.g., Hegel's concept of *determinate negation*). The main aim of this contribution is to tackle two related questions: *Is it possible to reduce the foundations of logic to the mere concept of incompatibility?* and *Does this reduction lead us to a specific logical system?* We conclude that the answers, respectively, are YES and a qualified NO (qualified in the sense that basing semantics on incompatibility does make some logical systems more natural than others, but without ruling out the alternatives.

A search for the bare bones of logic generally leads one to the relation of inference (or consequence). This way is explored meticulously by Koslow (1992). He defines an *implication structure* as, in effect,

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an ordered pair $\langle S, \vdash \rangle$, where S is a set and $\vdash \subseteq \text{Pow}(S) \times S$ fulfilling certain (relatively simple) restrictions. And obviously if we reduce incompatibility to inference, which is achievable by the well known *ex contradictione quodlibet* principle, we reach a logic based on incompatibility. The kind of logic flowing most straightforwardly from this setting is the intuitionist one.

However, there is also the approach taken by Brandom and Aker (2008), who have set up a logic based directly on incompatibility. They define an incompatibility structure as an ordered pair $\langle S, \perp \rangle$ such that S is a set and $\perp \subseteq \text{Pow}(S)$ (again fulfilling certain restrictions). The authors introduce logical operators in such a way that they reach classical logic.

Does this mean that inference 'naturally' leads to intuitionist logic, whereas incompatibility leads to the classical one? Myself, I have argued that it is indeed intuitionist logic that is *the* logic of inference (see Peregrin, 2008). However, this should not be read as saying that choosing inference as the fundamental logical notion predetermines us to have intuitionist logic, and that choosing incompatibility as such a notion perhaps predetermines us to have the classical one. I will argue instead that we can use incompatibility to lay the foundation of almost any imaginable kind of logic.

In what follows I first analyze the relationship between the two above mentioned approaches to logic based upon incompatibility, and the source of the difference between the ensuing logics; with the result that the source of the difference is not the choice of the fundamental notion itself, but rather certain collateral desiderata that Koslow, but not Brandom and Aker, poses for his logical system. Then I indicate that by generalizing the approach of Brandom and Aker, in a variety of ways, we can also reach other kinds of non-classical logic.

2 The framework

A *generalized inferential structure* (*gis*) will be the ordered triple $\langle S, \perp, \vdash \rangle$, where S is a set, $\perp \subseteq \text{Pow}(S)$ and $\vdash \subseteq \text{Pow}(S) \times S$. Which constraints should be placed on the notions of incompatibility and inference on this maximally general level? (It is clear that not any kind of set of sets of sentences can be reasonably seen as instantiating incompatibility, and that not every relation between sets of sentences and sentences can reasonably be called a relation of inference.)

Before introducing the constraints, a word about notation. The variables X, Y, Z will range over subsets of S, whereas the variables A, B, C will range over elements of S. $\perp X$ will denote that $X \in \perp$. $X \vdash A$ will denote that $\langle X, A\rangle \in \vdash$. We will write X, Y after $\perp$ or before $\vdash$ as a shortcut for $X \cup Y$, and A as a shortcut for $\{ A \}$. Hence, e.g. $\perp X, A$ expands to $X \cup \{ A \} \in \perp$. Now we can list the constraints we consider basic:

($\perp$) for every X, Y: if $\perp X$ and $X \subseteq Y$, then $\perp Y$

($\vdash$1) for every X, A: $X, A \vdash A$

($\vdash$2) for every X, Y, A, B: if $X, A \vdash B$ and $Y \vdash A$, then $X, Y \vdash B$

Let us adopt a further notational convention. Symbols that appear as "free" in the conditions of the above kind will be understood as universally quantified. Given this convention, we can shorten the above conditions to:

($\perp$) if $\perp X$ and $X \subseteq Y$, then $\perp Y$

($\vdash$1) $A, X \vdash A$

($\vdash$2) if $X, A \vdash B$ and $Y \vdash A$, then $X, Y \vdash B$

($\perp$) states that an incompatible set of sentences cannot be turned into a compatible one by addition of further sentences. This is the single constraint stipulated by Brandom and Aker. ($\vdash$1) states that if A belongs to X, then it is entailed by X. ($\vdash$2) says that the relation of consequence is transitive (if X entails every element of a set that entails A, then X entails A). The constraints ($\vdash$1) and ($\vdash$2) are stipulated by Koslow; they are tantamount to the so-called Gentzenian structural rules.[1]

It is sometimes useful to replace ($\vdash$1) by two other conditions:

[1]Gentzen (1934, 1935) introduced structural rules with which to characterize those relations of inference that he took to be 'standard'. In a slightly more contemporary manner, they can be summarized as follows:

$A \vdash A$	(*reflectivity*)
if $X, Y \vdash A$, then $X, B, Y \vdash A$	(*weakening* or *extension*)
if $X, A, A, Y \vdash B$, then $X, A, Y \vdash B$	(*contraction*)
if $X, A, B, Y \vdash C$, then $X, B, A, Y \vdash C$	(*permutation* or *exchange*)

Lemma 1 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\vdash$2) *holds. Then* ($\vdash$1) *holds if and only if* ($\vdash$1.1) *and* ($\vdash$1.2) *hold:*

($\vdash$1.1) $A \vdash A$;

($\vdash$1.2) *if* $X \vdash A$, *then* $X, B \vdash A$.

Proof. That ($\vdash$1) follows from ($\vdash$1.1) and ($\vdash$1.2) is obvious; conversely it is obvious that ($\vdash$1.1) follows from ($\vdash$1); so the only thing to show is that ($\vdash$1.2) follows from ($\vdash$1). Assume $X \vdash A$. As X, B entails all elements of X, $X, B \vdash A$ is yielded by the repeated application of ($\vdash$2). □

We may further consider constraints on the interplay of $\vdash$ and $\perp$. The most natural ones seem to be the following two:

($\perp\vdash$1) if $\perp X$, then $X \vdash A$

($\vdash\perp$1) if $X \vdash A$ and $\perp Y, A$, then $\perp Y, X$

The first says that an incompatible set entails everything (a version of the famous *ex falso quodlibet*, or, perhaps better, *ex contradictione quodlibet*); the second says that whatever is compatible with the antecedent of a consequence, cannot be incompatible with its consequent (Brandom and Aker call this condition—more precisely an equivalent one—*defeasibility*).

Adopting either of these conditions together with its converse allows us to reduce one of the two basic concepts to the other:

($\perp\vdash$2) if $X \vdash A$ for every A, then $\perp X$

($\vdash\perp$2) if $\perp Y, A$ implies $\perp Y, X$ for every Y, then $X \vdash A$

Thus, adopting ($\perp\vdash$1) plus ($\perp\vdash$2) is tantamount to reducing $\perp$ to $\vdash$, as treating a set of sentences as incompatible just in the case that it entails everything. From the other side, adopting ($\vdash\perp$1) plus ($\vdash\perp$2) is tantamount to reducing $\vdash$ to $\perp$, as treating a sentence as inferable

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if $X, A, Y \vdash B$ and $Z \vdash A$, then $X, Z, Y \vdash B$ (*cut*)

Within our framework, two of the conditions, namely *contraction* and *permutation*, are implicit to the assumption that inference is a relation between *sets* (rather than sequences) of sentences and sentences.

from a set of sentences just in the case that whatever is incompatible with the consequent is incompatible with its antecedent.

Let us call a gis *standard* iff it complies with ($\vdash$1), ($\vdash$2), ($\perp$), ($\vdash\perp$1), ($\vdash\perp$2), ($\perp\vdash$1), and ($\perp\vdash$2). Some of these constraints turn out to be superfluous.

Lemma 2 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\perp\vdash$1)*,* ($\perp\vdash$2)*,* ($\vdash$2)*, and* ($\vdash\perp$2) *hold. Then* ($\vdash\perp$1) *holds.*

Proof. Assume that $X \vdash A$ and $\perp Y, A$. Given ($\perp\vdash$1), it follows $Y, A \vdash B$ for every B. Given ($\vdash$2) we get $Y, X \vdash B$ for every B, and finally using ($\perp\vdash$2), we reach $\perp Y, X$. □

The situation is similar with ($\perp\vdash$1):

Lemma 3 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\perp$) *and* ($\vdash\perp$2) *hold. Then* ($\perp\vdash$1) *holds.*

Proof. Assume that $\perp X$. It follows from ($\perp$) that $\perp X, Y$ for every Y, hence it holds trivially that for every Y, A, if $\perp Y, A$, then $\perp Y, X$. Given ($\vdash\perp$2), $X \vdash A$. □

Corollary 4 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\perp$)*,* ($\perp\vdash$2)*,* ($\vdash$2)*, and* ($\vdash\perp$2) *hold. Then both* ($\vdash\perp$1) *and* ($\perp\vdash$1) *hold.*

Corollary 5 *$\langle S, \perp, \vdash \rangle$ is standard iff it complies with* ($\vdash$1)*,* ($\vdash$2)*,* ($\perp$)*,* ($\vdash\perp$2)*, and* ($\perp\vdash$2)*.*

Lemma 6 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\vdash\perp$1) *and* ($\vdash\perp$2) *hold. Then* ($\vdash$1.1) *and* ($\vdash$2) *hold; and* ($\vdash$1.2) *holds if* ($\perp$) *holds.*

Proof. As $\perp Y, A$ trivially implies $\perp Y, A$ for every Y and A, ($\vdash$1.1) holds for every A. To show that ($\vdash$2) holds, assume that $X, A \vdash B$ and $Y \vdash A$. In force of ($\vdash\perp$1), $\perp Z, B$ implies $\perp Z, X, A$ for every Z and $\perp Z, A$ implies $\perp Z, Y$ for every Z, hence especially $\perp Z, X, A$ implies $\perp Z, X, Y$ for every Z; and thus $\perp Z, B$ implies $\perp Z, X, Y$ for every Z. Hence, in force of ($\vdash\perp$2), $X, Y \vdash B$. Now assume that ($\perp$) holds and that $X \vdash A$. According to ($\vdash\perp$1), $\perp Z, A$ implies $\perp Z, X$ for every Z, and hence it also implies $\perp Z, X, B$ for every Z. Hence $X, B \vdash A$. □

Lemma 7 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\perp\vdash$1) *and* ($\perp\vdash$2) *hold. Then* ($\perp$) *holds if* ($\vdash$1.2) *holds.*

Proof. Assume ($\vdash$1.2) and assume $\perp X$ and $X \subseteq Y$. Then, according to ($\perp\vdash$1), $X \vdash A$ for every A, and hence, according to ($\vdash$1.2), $Y \vdash A$ for every A. Hence according to ($\perp\vdash$2), $\perp Y$. □

Corollary 8 *Let $\langle S, \perp, \vdash\rangle$ be a gis for which* ($\perp\vdash$1), ($\perp\vdash$2), ($\vdash\perp$1), ($\vdash\perp$2) *hold. Then* ($\perp$) *holds iff* ($\vdash$1.2) *holds.*

Given a gis, we can also define a *new* variant of incompatibility in terms of inference; and a new variant of inference in terms of incompatibility:

$$\vartriangle X =_{\text{Def.}} \text{for every } A, X \vdash A$$
$$X \vartriangleright A =_{\text{Def.}} \text{for every } Y,\ \text{if } \perp Y, A \text{ then } \perp Y, X$$

For a general gis, there is, of course, no guarantee that these new versions will coincide (or even be similar to) the original ones. However, suppose that the $\vdash$ which serves as the basis for the definition of $\vartriangle$ is already reducible to the original $\perp$—hence suppose that ($\vdash\perp$1) and ($\vdash\perp$2) hold. Then the definition of $\vartriangle$ in terms of $\vdash$ as if 'undoes' the definition of $\vdash$; and the two pairs of operators coincide.

This means that given ($\vdash\perp$1) and ($\vdash\perp$2), $\vartriangle$ (trivially) coincides with $\perp$; and analogously for $\vartriangleright$ and $\vdash$. Combining this trivial result with the claim of Corollary 4, we get:

Theorem 9[2] *Let $\langle S, \perp, \vdash\rangle$ be a gis for which* ($\perp$), ($\perp\vdash$2), ($\vdash$2), *and* ($\vdash\perp$2) *hold. Let $\vartriangle$ and $\vartriangleright$ be defined as above. Then $\vartriangle$ coincides with $\perp$ and $\vartriangleright$ coincides with $\vdash$.*

In this case we can say that $\perp$ and $\vdash$ 'fit together' in the sense that one can be reconstructed from the other.

3 Incompatibility vs. inference

Let us now turn our attention to logical operators; we will restrict ourselves just to two of them, namely negation and conjunction.

It seems that the minimal requirements that must be put on negation are the following:

[2] Half of this theorem was proved by Brandom and Aker under the name of the *Representation theorem (of consequence relations by incompatibility relations)*. What they proved was that $\vartriangleright$ coincides with $\vdash$, given ($\vdash$2) and ($\vdash\perp$2), which they call, respectively, *general transitivity* and *defeasibility*.

($\neg K1$) $\perp A, \neg A$

($\neg K2$) if $\perp A, B$, then $B \vdash \neg A$

These constraints stipulate that negation of A is its *minimal incompatible*: ($\neg 1$) states that $\neg A$ is incompatible with A, whereas ($\neg 2$) states that any other sentence incompatible with A is inferable from B. If we reduce incompatibility to inference, i.e. accept ($\perp\vdash 1$) and ($\perp\vdash 2$), we get:

($\neg K1'$) $A, \neg A \vdash B$

($\neg K2'$) if $A, B \vdash C$ for every C, then $B \vdash \neg A$

This gives us a natural characterization of negation in terms of inference. The consequences of this stipulation were investigated by Koslow; it leads to the intuitionist kind of negation.

Let us now consider a slight generalization of Koslow's definition more suitable for our purposes (the first condition stays the same):

($\neg 1$) $\perp A, \neg A$

($\neg 2$) if $\perp A, X$, then $X \vdash \neg A$

In his exposition of the character of physical laws, Feynman (1985) indicates how physical laws can be brought to the common denominator of a *minimum principle*; and the Koslowian approach to logic can be seen as the way of reducing logical operators (and the laws governing them) to a similar common denominator: all of them mark minima and maxima of functions defined in terms of inference. (This is not Koslow's invention, it goes back to Hertz and Gentzen, but Koslow has treated it more systematically.) However, if we now abandon this program and simply seek for any reasonable assortment of rules constitutive of negation in terms of inference (plus possibly incompatibility), we can think about a constraint that is dual to ($\neg 2'$):

($\neg 3$) if $\perp \neg A, X$, then $X \vdash A$

This constraint stipulates that the negation of A is a sentence whose *minimal incompatible* is A. It is important that this stipulation entails the law of double negation, which distinguishes classical from intuitionist logic.

Lemma 10 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* $(\neg 1)$ *and* $(\neg 3)$ *hold. Then* $\neg\neg A \vdash A$ *for every* A.

Proof. According to $(\neg 3)$ it is the case that if $\neg A \perp \neg\neg A$, then $\neg\neg A \vdash A$; and $\neg A \perp \neg\neg A$ is an instance of $(\neg 1)$. □

Brandom and Aker's definition of negation in terms of incompatibility is the following:

$(\neg B1)$ if $\perp X, \neg A$, then $X \vdash A$;

$(\neg B2)$ if $X \vdash A$, then $\perp X, \neg A$,

where $\vdash$ serves as a shortcut for "for every $Y \subseteq S$, if $\perp Y, A$, then $\perp Y, X$"; in other words we assume $(\vdash\perp 1)$ and $(\vdash\perp 2)$. What is the relationship between this definition and the above one? The answer is provided by the following two theorems:

Theorem 11 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* $(\vdash\perp 1)$, $(\neg 1)$, *and* $(\neg 3)$ *hold. Then* $(\neg B1)$ *and* $(\neg B2)$ *hold.*

Proof. As $(\neg B1)$ coincides with $(\neg 3)$, the only thing to prove is $(\neg B2)$. Hence assume $X \vdash A$. Then, according to $(\vdash\perp 1)$, it is the case that for every Y it holds that if $\perp Y, A$, then $\perp Y, X$. Hence especially if $\perp \neg A, A$, then $\perp \neg A, X$. Hence given $(\neg 1)$, we have $\perp \neg A, X$. □

Theorem 12 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* $(\vdash 1)$, $(\perp 1)$, $(\perp\vdash 1)$, $(\perp\vdash 2)$, $(\neg B1)$ *and* $(\neg B2)$ *hold. Then also* $(\neg 1)$, $(\neg 2)$, *and* $(\neg 3)$ *hold.*

Proof. We have already noted that $(\neg 3)$ coincides with $(\neg B1)$; and as $(\vdash 1)$ implies $A \vdash A$, $(\neg B2)$ yields us $(\neg 1)$; hence the only thing to prove is $(\neg 2)$. To prove it, assume that $\perp A, B$ and further assume that $\perp Y, \neg A$. Then, in force of $(\perp 1)$, $\perp A, B, Y$, and, in force of $(\neg B1)$, $Y \vdash A$. Thus, in force of $(\vdash\perp 1)$, $\perp Y, B, Y$, i.e. $\perp Y, B$. Hence for any arbitrary Y, if $\perp Y, \neg A$, then $\perp Y, B$, which yields $X \vdash \neg A$ via $(\vdash\perp 1)$. □

It follows that in a standard gis, $(\neg B1)$ and $(\neg B2)$ are equivalent to $(\neg 1)$, $(\neg 2)$ and $(\neg 3)$. Hence the fact that Koslow's approach leads to intuitionist negation, whereas Brandom's leads to the classical one, does not mirror any inherent difference between inference and incompatibility; the two approaches diverge because Koslow (like Gentzen) lays down specific constraints on how a feasible inferential definition of a logical constant ought to look, and that these constraints are fulfilled by ($\neg$1) plus ($\neg$2), not, however, by ($\neg$1), ($\neg$2) and ($\neg$3).

Now let us turn our attention to conjunction. Koslow's way of introducing it is as the upper bound (or supremum, if you prefer):

($\wedge K1$) $A \wedge B \vdash A$

($\wedge K2$) $A \wedge B \vdash B$

($\wedge K3$) if $X \vdash A$ and $X \vdash B$, then $X \vdash A \wedge B$

Lemma 13 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\wedge K1$)*,* ($\wedge K2$)*,* ($\vdash$1)*,* ($\vdash$2) *hold. Then* ($\wedge K3$) *is equivalent to*

($\wedge K3'$) $A, B \vdash A \wedge B$.

Proof. Assume ($\wedge K3$). As ($\vdash$1) yields us $A, B \vdash A$ and $A, B \vdash B$, we get $A, B \vdash A \wedge B$ by means of ($\vdash$2). Now assume ($\wedge K3'$) and assume $X \vdash A$ and $X \vdash B$. We get ($\wedge K3$) by means of ($\vdash$2). □

Brandom and Aker's definition of conjunction is the following:

($\wedge B1$) if $\perp X, A \wedge B$, then $\perp X, A, B$

($\wedge B2$) if $\perp X, A, B$, then $\perp X, A \wedge B$

Theorem 14 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\wedge K1$)*,* ($\wedge K2$)*,* ($\wedge K3$)*, and* ($\vdash\perp$1) *hold. Then* ($\wedge B1$) *and* ($\wedge B2$) *hold.*

Proof. To prove ($\wedge B1$), assume $X \perp A \wedge B$. We obtain $\perp X, A, B$ by means of ($\vdash\perp$1) and ($\wedge K3$). To prove ($\wedge B2$), assume $\perp X, A, B$. Using ($\vdash\perp$1) and ($\wedge K1$) we obtain $\perp X, B, A \wedge B$; and using ($\vdash\perp$1) and ($\wedge K2$) we further obtain $\perp X, A\wedge B, A\wedge B$; hence $\perp X, A\wedge B$. □

Theorem 15 *Let $\langle S, \perp, \vdash \rangle$ be a gis for which* ($\wedge B1$)*,* ($\wedge B2$)*,* ($\perp$1)*,* ($\vdash\perp$2) *hold. Then* ($\wedge K1$)*,* ($\wedge K2$)*, and* ($\wedge K3$) *hold.*

Proof. To prove ($\wedge K1$), assume $\perp X, A$. Using ($\perp$1) we get $\perp X, A, B$. ($\wedge B2$) then yields us $\perp X, A \wedge B$. Hence if $\perp X, A$, then $\perp X, A \wedge B$; and using ($\vdash\perp$2) we get ($\wedge K1$). The proof of ($\wedge K2$) is straightforwardly analogous. To prove ($\wedge K3$), assume $\perp X, A \wedge B$. With the help of ($\wedge B1$) we obtain $\perp X, A, B$. Now we get $A, B \vdash A \wedge B$ using ($\vdash\perp$2). □

This ultimately clarifies the relationship between Koslow's inferential logic and Brandom and Aker's incompatibility logic, and explains why the former leads to the intuitionist, while the latter to classical logic. The distinctness of the outcomes do not stem from any inherent differences between inference and incompatibility (indeed the two concepts are two sides of the same coin); rather, it stems from the discipline that Koslow, in contrast to Brandom and Aker, adds to his inferential foundations.

4 Beyond classical and intuitionist logic

We have seen that basing logic on incompatibility and/or inference naturally leads us to either classical, or intuitionist logic. Is it possible to conceive of basing other kinds of logic, such as relevant logic or modal logic, on incompatibility and/or inference? The case of relevant logic is straightforward: we know that to be able to introduce it, we need first to eliminate:

($\vdash$1.2) if $X \vdash A$, then $X, B \vdash A$,

which, as we saw in Corollary 8, is equivalent to ($\perp$). Hence all it takes to prepare the ground for the relevantist version of logic within the framework of incompatibility logic is to retract ($\perp$).

To accommodate other kinds of substructural logics, such as linear logic, we must interfere deeper with our foundations. It is well known that linear logic requires us to see inference as a relation not between subsets of S and elements of S, but rather between multisets of the elements of S and elements of S, which allows us to discard the structural rule of contraction. And we can go further and replace multisets with sequences, which gives us the opportunity to discard permutation, thus allowing for logics for which the order of premises—and, as the case may be, conclusions—is significant, e.g. some dynamic logics. The situation is much the same as for incompatibility logic: to make room for linear logic we must consider incompatibility not as a property of sets, but rather of multisets (so that, for example, the multiset $\{ A, A, B \}$ may be inconsistent though $\{ A, B \}$ is consistent; or *vice versa*) and we might further consider it as a property of sequences.

Of course, there is a conceptual question concerning the extent to which it makes sense to consider incompatibility as a property of sequences of sentences rather than its sets (just as there is the conceptual question of how far it makes sense to consider inference as a relation between sequences of sentences, rather than between sets of sentences and sentences). But at least some reasons appear to be available: we know, for example, that a collection of sentences presented in one order may make up a consistent story, whereas the same sentences in a different order may not.

How is it with modal logic? Brandom and Aker introduced a natural definition of the necessity operator based on incompatibility:

($\Box$) $\perp X, \Box A$ iff $\perp X$ or there is an Y such that not $\perp X, Y$ and not $Y \vdash A$.

They show that this definition of incompatibility leads to the simplest modal logic S5. Can we have different modal logics based on incompatibility? In principle, surely we can, provided that we add some surplus ingredient corresponding to the relation of equivalence on Kripkean models. Elsewhere (see Peregrin, 2010) I have shown how we may reach logic B in this way.

5 Conclusion

It is possible to base logic solely on the concept of incompatibility; and in fact it does not restrict us in any substantial way w.r.t. the kind of logic we want to have. Brandom and Aker's elegant way of establishing it leads us to classical logic and further possibly to S5. (In contrast to logic based on inference that appears to naturally yield intuitionist logic.) In this sense, these logics may appear to be "natural" from the viewpoint of incompatibility logic; however, if naturalness is not what we are after, nothing prevents us from erecting almost any kind of logic on incompatibility foundations.

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