

CONSEQUENCE & INFERENCE

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1. Inference as an explication and as a counterpart of consequence

Logic is usually considered to be the study of logical consequence – of the most basic laws governing how a statement's truth depends on the truth of other statements. Some of the pioneers of modern formal logic, notably Hilbert and Carnap, assumed that the only way to get hold of the relation of consequence was to reconstruct it as a relation of inference within a formal system built upon explicit inferential rules. Even Alfred Tarski in 1930 seemed to foresee no kind of consequence other than one induced by a set of inference rules: "Let A be an arbitrary set of sentences of a particular discipline. With the help of certain operations, the so-called *rules of inference*, new sentences are derived from the set A , called the *consequences of the set A* . To establish these rules of inference, and with their help to define exactly the concept of consequence, is again a task of special metadisciplines; in the usual terminology of set theory the schema of such definition can be formulated as follows: The set of all *consequences of the set A* is the intersection of all sets which contain the set A and are closed under the given rules of inference." (p. 63)

Thereby also the concept of truth came to be reconstructed as inferability from the empty set of premises. (More precisely, this holds only for non-empirical, necessary truth; but of course logic never set itself the task of studying empirical truth.) From this viewpoint, logic came to look as the enterprise of *explication of consequence in terms of inference*.

This view was soon shattered by the incompleteness proof of Kurt Gödel and by the arguments of Tarski himself, which seemed to indicate that inference can never be a fully satisfying explication of consequence, and that hence we must find a more direct way of dealing with consequence. And model theory, established by Tarski, appeared to be the requisite tool. Consequence and inference thus started to appear as two things to be confronted and one of the main issues of logic became the investigation of the results of various instances of the confrontations – various kinds of soundness and completeness theorems. However, as Bencivenga (1999, 16) duly points out, "completeness theorems are a dime a dozen, but there is hardly any discussion of what it means to prove such a theorem."

2. Consequence

What is consequence? When is a statement a consequence of other statements? As we have already noted, consequence is generally understood as truth-preservation: A is a consequence of A_1, \dots, A_n iff A is true whenever A_1, \dots, A_n are true. However, what does the "whenever" amount to here? It appears to be a matter of a universal quantification over some universe of cases – so what are these cases supposed to be?

On first sight, it seems that the "whenever" has to mean simply "in all possible circumstances". This construal is plausible for empirical sentences, but it is important to realize that for non-empirical, esp. mathematical statements it would identify consequence with material implication: thus, any true mathematical statement would be a consequence of any mathematical statements whatsoever and any mathematical statement whatsoever would be a consequence of every false mathematical statement. This sounds quite implausible; and hence it is worth paying attention to other explications of the relevant "whenever".

An alternative idea can be traced back to Bernard Bolzano (1837)¹, who proposed, in effect, that the "whenever" should be construed as "under every interchange of certain parts of expressions in question". (Bolzano's direct target was *analyticity*, which, however, is interdefinable with consequence, at least within 'usual' languages²: A is a consequence of A_1, \dots, A_n iff the sentence *if A_1, \dots, A_n , then A* is analytic.) Hence the generality alluded to here is not a factual one, consisting in considering possible states of affairs, but rather a linguistic one, consisting in considering possible substitutions of expressions for other expressions (and hence substitutional variants of the relevant sentences).

Note that this latter generality partly emulates the former one. Consider the sentence:

Dumbo is an elephant.

This sentence is true w.r.t. some circumstances (those in which Dumbo is an elephant) and is false w.r.t. others (those in which Dumbo is not an elephant). But these kinds of circumstances can be seen as 'emulated' by kinds of interpretations – namely the first by interpretations interpreting "Dumbo" as the name of an (actual) elephant and the second by interpretations interpreting "Dumbo" as the name of something else.

The problem is how to draw a line between that part of vocabulary we should hold fixed and that which we are to vary. Take the following obvious instance of consequence:

Dumbo is an elephant

Every elephant is gray

There is something that is gray

Intuitively, this is an instance of consequence for it is inconceivable that Dumbo is an elephant, every elephant is gray and yet there exists nothing that is gray. To this there corresponds the fact that substituting names of other entities for "Dumbo" and names of other sets of entities for "elephant" and "gray" we cannot make the sentences in the antecedent true while that in the consequent at the same time false. Hence it might seem that what we should

¹ Or perhaps even much farther back – see King (2001).

² Languages having a logical connective of the kind of *if ... then ...*. Obviously, this can be taken for granted for any natural language, and it is also fulfilled by common logical languages. But, of course, there may be artificial languages (or 'languages?') lacking any such connective.

vary is simply empirical vocabulary – however, we have already noted that this would make it impossible to make any nontrivial sense of consequence within mathematics.

Moreover, take the following similar case of consequence:

Dumbo is an elephant

Every elephant is an animal

There is something that is an animal

Could there be a situation in which an elephant is not an animal? Hardly – if something were not an animal, we could not reasonably call it an elephant. (Note that it would be wrong to reason: 'if "elephant" represented not elephanthood, but, say trainhood, then elephants would *not* be animals' – as Abraham Lincoln has already observed, a dog would keep to have four legs even if we decided to call his tail a leg.) Hence it seems that this instance of consequence is tantamount to:

Dumbo is an elephant

There is something that is an animal

And it is clear that the only part which can be safely varied here is the name "Dumbo" – the identity of the predicates "elephant" and "animal" becomes substantial. (In other words, it seems we must acknowledge what Sellars, 1948, called *material* instances of consequence.) Hence precisely *which* kinds of expressions should we require to be varied?

Bolzano seemed to imply that analyticity emerges wherever there is a *salva veritate* variation of *any* part of a sentence. However, this cannot be correct. Take the sentence:

Dumbo is sleeping and nothing that George Bush thinks of it can change it.

It would seem that if it is indeed the case that Dumbo is sleeping, then the truth of the sentence cannot be affected by substituting in it a name other than George Bush – but should we see the sentence therefore as analytic?

A possible way of avoiding this problem is to narrow our focus and concentrate on merely *logical* consequence – consequence 'in force of' logical vocabulary alone. In a sense, this does *not* narrow down the scope of instances we can consider, for as Frege (1879) already noticed, *B's* being a consequence of *A* can also be understood as its being a *logical* consequence of *A* and *if A, then B*.³ This, admittedly, institutes no *sharp* boundary, for there is no strict criterion for distinguishing between logical and extralogical words; however, loose criteria, such as topic-neutrality, are at hand.

³ This led many logicians to the conclusion that the former case of consequence is only a disguised version of the latter – that the step from the former to the latter amounts to disclosing a covert presupposition. However, as Sellars (1953) pointed out, it may be more plausible to see the situation completely in reverse – to see logical consequences as explicative of the material ones.

But there is also a more serious shortcoming of the Bolzanian method, noted already by Bolzano himself: it makes consequence depend on the contingent fact of the richness of the language in question (something might cease to be an instance of consequence by the introduction of a new expression enabling us to articulate a counterexample.). We can call this the problem of the (possible) "poverty of language". Bolzano avoided it by basing his definition on an 'ideal' language, language *per se*, which as such cannot lack anything.

Bolzano's modern successors, in particular Alfred Tarski (1936), avoided this recourse to an ideal language by offering an alternative solution. The point of interchanging expressions within Bolzano's approach is in gaining other sentences with the same structure as the original one; and clearly the same thing could be effected by varying the meanings of the original expressions. (Replacing the meaning of "Dumbo" by that of "Batman" has clearly the same effect as replacing the word "Dumbo" itself by the word "Batman".) Hence we could replace Bolzanian substitutions by *interpretations*: assignments of appropriate kinds of objects to expressions as their denotations. This has the advantage that we solve the problem of the "poverty of language" without having to presuppose some such entity as an ideal language *per se*.

3. Inference

The term "inference" is in itself problematic, because of its multiple ambiguities. First, there is inference in the sense of the acts of inferencing carried out by concrete people in concrete circumstances. This is not a topic we are usually interested in when doing logic – it is rather a matter of cognitive psychology⁴. Let us use the term "inference₁" for inference conceived in this sense (perhaps we could also use the term "reasoning"; and then we would have to conclude, together with Harman, 2002, that *logic is not a theory of reasoning*).

Second, there is inference in the sense of the relationship of *correct inferability* – as the result of the fact that we often hold the acts of inferencing (reasoning) people undertake for right or wrong. Inference in this sense can be seen as a relation between (sets of) sentences and sentences, which is, however, in the sense just mentioned intimately connected to people's doings. We will refer to it as "inference₂".

Third, there is inference in the sense of an arbitrarily defined abstract relation, usually generated by a system of inferential rules related to an artificial language. This last relation can be used as an explication of the previous one; but it might also be brought into being by purely mathematical interests. We will refer to inference in this sense as "inference₃".

Confusing these three senses of "inference" often leads to fatal perplexities within philosophy of logic; and, unfortunately, it is rather common. Let me mention briefly two very common kinds of confusion, which I have discussed in greater detail elsewhere.

⁴ Criticism of logic based on the very assumption that this *is* what logic is after (see, e.g., Perkins, 2002) is, however, perennial.

Firstly, there is the confusion of inference₁ with inference₂. As an example we can take the recent discussion of Brandom's (1994) inferentialism (analyzed in detail in Peregrin, ms.). Fodor and LePore (1993), along with other philosophers, argue that inference cannot be a basis for meaning – however, the trouble is that while Brandom construes the term "inference" as inference₂, Fodor and LePore base their critique on a construal amounting to inference₁. Needless to say, given this, both sides cannot but talk past each other.

Secondly, in Peregrin (2006a) I pointed out, in effect, that there is also a common confusion between inference₂ and inference₃, which leads to the claim that inference is a purely syntactic matter. This surely holds about inference₃, but not about inference₂. While inference₃ is based on a wholly arbitrary set of rules; the rules of inference₂ are, by definition, truth-preserving. I argued that it is this kind of confusion which leads to the arguments that computers cannot think because they can have only syntax and not semantics (see Searle, 1984): computers can clearly have *rules* (and hence inference₂) which, if truth-preserving, are more than syntax (inference₃).

Sellars (1953) duly points out that the concept of inference as put forward by Carnap (1934) unsuccessfully tries to ride all the three horses at the same time. Carnap's inference (as well as the inference of many contemporary logicians) is in fact inference₃, and Carnap makes a deep point of the fact that the relation is fully arbitrary. On the other hand, he refers to it as the relation of *derivability*, which, Sellars points out, alludes to the fact that it expresses what *is permitted to be derived* (which would be appropriate for inference₂). Moreover, as what is or is not permitted are human actions, viz. those of inferring, it further alludes to the classification of human inferential performances, which are a matter of inference₁. But Carnap pays no attention to any constraints which would be implied for his definition of inference by any normative considerations or empirical studies of human inferential activities. As a result, Sellars concludes that "Carnap's claim that he is giving a definition of 'directly derivable in *S*' is a snare and a delusion" (329).

Keeping these distinctions in mind, what can we say about the relation between consequence and inference? Consequence has very little to do with inference₁. Consequence is an objective matter (what follows from what does not depend on whether I or you believe it to follow⁵), whereas inference₁ is purely individual. The fact that somebody announces that he will soon cease seeing me, from which I infer that he is about to kill me has very little to do with consequence; and the fact that somebody is disposed to infer "This is a fish" from "This is a cat" does not undermine the fact that (in English) "This is a fish" is *not* a consequence of "This is a cat".

On the other hand, consequence and inference₂ can be seen as simply two sides of the same coin (at least when we restrict ourselves to a finite number of premises). The reason is that *to be correctly inferable from* is nothing else than *to be a consequence of* – on the one hand we can infer *B* from *A* if the truth of *A* guarantees the truth of *B*; on the other hand, we

⁵ Of course it depends on the existence of the shared language and thereby on certain 'beliefs' of members of the relevant linguistic community. However, this does not make consequence not objective – at least it is surely no less objective than chess or NATO or money, which all also depend on certain 'beliefs' of people.

can do so only if there is such a guaranty. Hence insofar as inference₃ is a suitable tool for the explication of inference₂, it is a suitable tool for the explication of consequence.

Elsewhere (Peregrin, 1995) I called this kind of explication *criteria reconstruction*; and I still find this term very instructive. What is going on here is that a concept whose extension is not delimited by an explicit rule (but rather only in terms of practical know-how) is associated with a *criterion*. The explicit criterion cannot exactly replicate the boundaries of the implicitly delimited extension – for the extension does not have exact boundaries, it is more or less fuzzy.

4. The cleft

Let us review the usual reasons for asserting the existence of an unbridgeable cleft between consequence and inference. I cite the most explicit argument due to Tarski in full (cf. also my discussion of this passage in Peregrin, 1997):

Schon vor mehreren Jahren habe ich ein Beispiel, übrigens ein ganz elementares, einer derartigen Theorie gegeben, die folgende Eigentümlichkeit aufweist: unter den Lehrsätzen dieser Theorie kommen solche Sätze vor wie:

A₀. 0 besitzt die gegebene Eigenschaft E,

A₁. 1 besitzt die gegebene Eigenschaft E,

u.s.w., im allgemeinen alle speziellen Sätze der Form:

A_n. n besitzt die gegebene Eigenschaft E,

wobei 'n' ein beliebiges Symbol vertritt, das eine natürliche Zahl in einem bestimmten (z. B. dekadischen) Zahlensystem bezeichnet; dagegen läßt sich der allgemeine Satz

A. Jede natürliche Zahl besitzt die gegebene Eigenschaft E,

auf Grund der betrachteten Theorie mit Hilfe der normalen Schlußregeln nicht beweisen. Diese Tatsache spricht, wie mir scheint, für sich selbst: sie zeigt, daß der formalisierte Folgerungsbegriff, so wie er von den mathematischen Logikern allgemein verwendet wurde, sich mit dem üblichen keineswegs deckt. Inhaltlich scheint es doch sicher zu sein, daß der allgemeine Satz A aus der Gesamtheit aller speziellen Sätze *A₀, A₁, ..., A_n, ...* im üblichen Sinne folgt: falls nur alle diese Sätze wahr sind, so muß auch der Satz A wahr sind⁶.

⁶„Some years ago I gave a quite elementary example of a theory which shows the following peculiarity: among its theorems there occur such sentences as: *A₀. 0 possesses the given property P, A₁. 1 possesses the given property P,* and, in general, all particular sentences of the form *A_n. n possesses the given property P,* where 'n' represents any symbol which denotes a natural number in a given (e.g. decimal) number system. On the other hand the universal sentence: *A. Every natural number possesses the given property P,* cannot be proved on the basis of the theory in question by means of the normal rules of inference. This fact seems to me to speak for itself. It shows that the formalized concept of consequence as it is generally used by mathematical logicians, by no means coincides with the common concept. Yet intuitively it seems certain that the universal sentence A

Can we say that this argument shows that inference and consequence can never coincide? Well, there is a rather shallow sense in which this is obvious: while it does not appear to make sense to talk about inference with an infinite number of premises (insofar as the inference relation amounts to the correctness of human inferences and no human could actually handle an infinite number of sentences), there seems no reason not to consider consequences with infinite numbers of premises (this follows besides other from the fact that it seems reasonable to admit that if something follows from some premises, then it also follows from any *more* premises).

So the *nontrivial* difference between consequence and inference would obtain only there existed a statement which followed from an infinite set of premises without following from any of its finite subsets – in other words if consequence were not *compact*. And it seems that Tarski's example shows that this is indeed the case.

A parallel case against the identifiability of consequence with inference is entailed by Gödel's incompleteness proof. One of the direct consequences of the proof is usually taken to be the fact that for any axiom system of arithmetic, there is an arithmetical sentence which is true, but not provable within the system (intuitively, it is the sentence which 'codifies' the claim that it itself is unprovable). Moreover, as the truth of mathematical sentences does not depend on states of the world, this sentence must be true necessarily, i.e. must be entailed by the empty set. However, it is not inferable from the empty set – hence again, inference would seem to lag behind consequence.

Both Tarski's and Gödel's case concern, at least *prima facie*, arithmetic, hence not directly *logical* consequence. However, while in the former case this seems to be essential (though Edwards, 2003, argues that what Tarski had in mind *was* a specific, disguised case of *logical* consequence), the latter is easily convertible to the domain of pure logic. It would imply that the undecidable sentence is a logical consequence of the axioms of arithmetic (and within second-order logic, where arithmetic is *finitely* axiomatizable, it thus follows logically from a *finite* number of sentences) without being inferable from them⁷.

These facts made the majority of modern logicians accept Tarski's proposal to investigate consequence (and truth) not via inference (proof theory), but via *model theory* – Tarski's (1936) widely accepted explication of consequence is A is a consequence of A_1, \dots, A_n iff every model (i.e. verifying interpretation) of A_1, \dots, A_n is also a model of A (see also Priest, 1999). This is a reconstruction of the concept of consequence which is also 'criterial', but in a weaker sense: it not only does not allow us to always decide whether a given sentence is a consequence of some other given sentences, but it not even allows us to generate all instances of consequence. This also opened up the problem of the relationship between consequence and inference for individual

follows in the usual sense from the totality of particular sentences $A_0, A_1, \dots, A_n, \dots$. Provided all these sentences are true, the sentence A must also be true.“

⁷ It would be argued that the truth of Gödel's sentence does not follow from the axioms of arithmetic, for it is not true in all models of the axioms. However, this objection turns on the first-order regimentation of arithmetic (which allows for non-standard models), which cannot be equated with arithmetic as such. Within second-order arithmetic, there are no models of the axioms in which Gödel's sentence would be false.

calculi as an important research theme – to what extent can we turn the 'loosely criterial', model-theoretic delimitation of consequence into the 'strictly criterial', proof-theoretic one?

5. The bridge of meaning

The cleft thus opened between model theory and proof theory paved the way into vast new realms of interesting mathematics; however, should we read it as showing that consequence is wholly independent of inference, being only – in a better or worse way – mimicked by inference? There is a substantial argument against this conclusion, an argument turning on the concept of meaning.

Why is a sentence, say "There is something that is gray", a consequence of other sentences, say "Every elephant is gray" and "Dumbo is an elephant"? This must clearly be in virtue of the *meaning* of all the sentences involved. Hence how is it that the sentences do have the meaning they have? Again, this must be a matter of the way we, as the members of the relevant community, treat them – sounds and inscriptions do not mean anything without our endeavor. Hence, what have we done to the sentences to have made them mean what they do?

There are two popular kinds of answers to this question, both backed by a host of advocates. The first is that our words mean something because we let them *stand for* this something; the second is that they mean something because we *use* them in a certain way (see Peregrin, 2004). In this general form, these two answers do not seem to be intrinsically incompatible – perhaps we use words in such a way that they come to stand for something? But the usual elaborations of these two kinds of answers do lead to incompatible approaches.

The first standpoint standardly leads to the view that we make words meaningful in that we *conventionally establish* that they become *symbols* for various kinds of extralinguistic entities, which thus become their meanings. The second one leads to the view that meanings of words should be construed not as something represented by them, but rather as the very ways in which they are used, as their functions or roles within our linguistic transactions.

How would these two answers fare when answering the subsequent question about the origin and nature of consequence? It seems that the first must claim that any semantic relation between expressions, such as that of consequence between statements, cannot but be a mimic of a relationship between the entities stood for by them. One may, for example, want to claim that sentences entail other sentences because they stand for facts and some facts contain other facts – hence that the relation of consequence is a linguistic reflection of the nonlinguistic relation of containment.

Is this answer viable? Hardly. It would mean that "Dumbo is an elephant" entailing "Dumbo is an elephant or is a rhino" is the results of the following three facts:

- (i) that we have introduced of the former sentence as a name of a fact;
- (ii) that we have introduced the latter as a name of *another* fact (*entirely independently* of the first naming); and
- (iii) that the first of these facts happens to contain the second.

Needless to say, even if we disregard all problems connected to the concept of fact and admit that a sentence like "Dumbo is an elephant" can be reasonably seen as a name of a fact, it is hard to lend any credibility to a theory which assumes that the fact named by "Dumbo is an elephant" can be only empirically discovered to be part of the fact named by "Dumbo is an elephant or is a rhino".

It seems much more plausible to assume that whatever the meaning of "Dumbo is an elephant" is, if it gets combined with another sentence by means of "or", the meaning of the result derives from the meanings of the parts in a way determined by "or", and the determination consists especially in (though it is perhaps not reducible to) the consequential links between the complex sentence and the parts. Hence we move to the second paradigm of meaning: meaning consists in the way an expression is used, and the meaning of at least some sentences comes purely from the *inferential rules* which govern them. From this viewpoint, meanings are to be identified by the roles conferred on words by those very rules (just as the roles of chess pieces consists exclusively in from the fact that we treat them according to the rules of chess). And the rules which are basically relevant from the viewpoint of meaning are inferential rules – logical (and perhaps some other) words are governed exclusively by this kind of rules, whereas empirical words are governed by these rules together with different types of rules maintaining 'links between language to the world', i.e. stating in what circumstances it is correct to utter some sentences and perhaps what kinds of action are appropriate in response to an utterance.

If we agree with this, then conferring a meaning on a logical word is accepting a basic inferential pattern governing the usage of some sentences containing the word. Conferring the usual meaning on *and* is accepting that it is correct to infer *A and B* from *A* and *B*, and that it is correct to infer both *A* and *B* from *A and B*. Now, however, it seems that consequence must be entirely brought into being by inference: for consequence is a product of meaning and meaning is a product of inferential rules. Is this viable?

6. From an inferential pattern to a relation of inference

Note that the inferential pattern associated with a statement is not a list of everything inferable from that statement and everything from which the statement is inferable. Rather, it is a collection of the most basic inferences which are supposed somehow to "establish" many others. However, in what sense do they "establish" them? Take conjunction. The pattern governing it is:

$$\begin{aligned} A \wedge B & \vdash A \\ A \wedge B & \vdash B \\ A, B & \vdash A \wedge B \end{aligned}$$

and we take it to "establish" a number of other inferences, including:

$A \wedge B, C \dashv\vdash A$
 $(A \wedge B) \wedge C \dashv\vdash A$
 $A \wedge B \dashv\vdash B \wedge A$
 etc.

What kind of inferences are these?

Let us first sharpen our terminology. Where S is a class of statements, an *inference over S* will be an ordered pair of a finite set of elements of S and an element of S . An *inferential relation over S* will be a set of inferences over S . What we are now looking for is a characterization of how a basic inferential relation ("inferential pattern") R naturally leads to ("establishes") a wider inferential relation R^* .

There are several ways of approaching this problem. The most common characterization is that R can be taken to underlie the wider relationship of "provability in terms of R ". Hence, we may say, $\langle M, s \rangle \in R^*$ iff there is a sequence s_1, \dots, s_n of statements such that $s_n = s$ and each s_i is either an element of M or there is an $\langle N, s_i \rangle \in R$ such that $N \subseteq \{s_1, \dots, s_{i-1}\}$.

We may reach a variant of this answer by characterizing the general relationship between R and R^* in terms of closure conditions: if we find some operations on $Pow(S) \times S$ such that R^* is always the smallest superset of R closed to these operations. We may call these operations *metainferences*, for they infer inferences from inferences. And in fact we do not need to seek them, for they were disclosed long ago by Gentzen (1934) – they are nothing else than the structural rules of his sequent calculus. In other words, R^* is the smallest set of inferences which contains R and⁸

- (a) contains $\langle \{s\}, s \rangle$ for every s
- (b) contains $\langle N', s \rangle$ whenever it contains $\langle N, s \rangle$ and $N \subseteq N'$ ⁹
- (c) contains $\langle N \cup (N' \setminus s), s' \rangle$ whenever it contains $\langle N, s \rangle$ and $\langle N', s' \rangle$ and $s \in N'$

It is very plausible to assume that it is via these closure conditions that an inferential pattern establishes a wider class of valid inferences than those constituting the pattern. (Though there are also well known objections – like the relevantists' rejection of (b) etc.) However, might there not be *an additional*, less obvious, way in which an inferential pattern can establish a *still wider*, 'quasiinferential' relation?

⁸ As we treat inferences as pairs each consisting of a *set* of statements and a statement, we do not need the contraction and permutation.

⁹ This rule may, or may not, be restricted to finite sets of premises – in the latter case it leads us to the abandonment of the realm where we can speak about inference straightforwardly; however, as it cannot escape compactness, this abandonment is merely trivial.

7. Consequence is quasiinference ...

In fact, this idea is not without precedent. In his *Logical syntax of language* Rudolf Carnap distinguished between *Beweisbarkeit* (*provability*) and *Folgerung* (*consequence*), both being established by the axioms and rules of the relevant system. The difference between them was precisely the one hinted at above – the way leading from the system to consequence is a generalization of the one leading from it to provability. In the first of the two model languages presented in Carnap's book, the distinction consists merely in the fact that a general number-theoretic sentence is a consequence of all its number-theoretic instances, though it is not provable from them. (In the second one he replaced his 'quasiinferentialist' approach to consequence with a wholly different one, basing his definition of consequence on an almost model-theoretical definition of analyticity – see Coffa, 1991, Chapter 16, for a thorough discussion.) This amounts to the infinitary rule of inference (where N is the predicate delimiting natural numbers):

$$P(0), P(1), P(2), \dots \dashv\vdash \forall x(N(x) \rightarrow P(x)).$$

or, equivalently:

$$N(x), P(0), P(1), P(2), \dots \dashv\vdash P(x). \tag{\omega-1}$$

Thus, it seems that Carnap simply took the very same case which troubled Tarski and made it into a (quasi)inferential rule. In fact Tarski did consider the very same possibility. (Tarski, 1932, realized that the addition of this rule would align inference with consequence, but rejected considering the rule as a rule *of inference*, for it is "infinitistic". However, it seems to follow that at least for arithmetic he would see the 'quasiinference' constituted by the normal inference plus the ω -rule equivalent to his model-theoretically defined consequence.)

As I pointed out elsewhere (Peregrin, 2006b), there is also another available explanation for how an inferential pattern can be envisioned to establish a relation of consequence. Arguably, at least some inferential patterns can be seen as sorts of exhaustive lists, as exhaustive lists of either the principal grounds of a given sentence or its principal corollaries. Thus, putting forward:

$$\begin{aligned} A &\dashv\vdash A \vee B \\ B &\dashv\vdash A \vee B \end{aligned}$$

as a pattern might be read as giving a list of sentences from which $A \vee B$ follows, which is exhaustive in the sense that any C which also entails $A \vee B$ must entail either A or B .

This particular case is not interesting from the current viewpoint, as it establishes no new inferences or quasiinferences (though it does manage to pin down the meaning of \vee to classical disjunction which is impossible otherwise). However, in a similar way, putting

forward the induction scheme *as a self-contained inferential pattern* may be read as not only stating that 0 is a number and a successor of any number is a number, but also that nothing else is a number. This is to say that stipulating the pattern:

$$\begin{array}{l} \text{|-- } N(0) \\ N(x) \text{ |-- } N(x') \end{array}$$

we implicitly stipulate that nothing is a number unless it follows from these inferences, i.e. unless it either is 0, or is accessible from 0 by iteration of the successor operation. It might seem that this requirement can be expressed in terms of the stipulation that if $N(x)$ then either $N(0)$ or there is a y such that $x=y'$, but this is not the case as this would obviously be valid even in nonstandard models of arithmetic. What is needed to express the requirement in its full strength is the infinitary sequent (the notation is the usual Gentzenian one – 'if every item before |--, then at least one item thereafter'):

$$N(x) \text{ |-- } x=0, x=1, x=2, \dots \tag{\omega-2}$$

And it is clear that this is precisely what is needed to exclude the nonstandard models and hence pin down the model of the theory to the intuitive natural numbers.

Now it can be shown that $(\omega-1)$ and $(\omega-2)$ are equivalent (and hence that the assumption of the induction rule is the same assumption as the assumption of the validity of the ω rule). To do so, suppose first that $(\omega-1)$ holds and take $P(y)$ to be $x \neq y$:

$$N(x), x \neq 0, x \neq 1, x \neq 2, \dots \text{ |-- } x \neq x.$$

Using the Gentzenian rules of negation elimination, this yields

$$N(x), x=x \text{ |-- } x=0, x=1, x=2, \dots$$

and hence, as $x=x$ is valid, $(\omega-2)$.

Now suppose, conversely, that $(\omega-2)$ holds. As it is the case that

$$P(0), x=0 \text{ |-- } P(x),$$

we can use cut on this sequent and $(\omega-2)$ to obtain

$$N(x), P(0) \text{ |-- } P(x), x=1, x=2, \dots$$

In the same way we can use

$$P(1), x=1 \text{ |-- } P(x),$$

to further obtain

$$N(x), P(0), P(1) \dashv\vdash P(x), x=2, \dots$$

and ultimately

$$N(x), P(0), P(1), P(2), \dots \dashv\vdash P(x),$$

i.e. (ω -1). (Note that as we consider the list on the sides of $\dashv\vdash$ as representing *sets*, contraction is automatic.)

8. ... but quasiinference is not really inference!

Consider (ω -1) once more. It is clearly not a case of logical consequence – it says that *Every natural number is P* follows from $\{n \text{ is } P\}_{n=1, \dots, \infty}$. But it is clear that *Every natural number is P* does *not* follow from $\{n \text{ is } P\}_{n=1, \dots, \infty}$ logically; for the logical form of the argument would be

$$P(T_1), P(T_2), \dots \dashv\vdash \forall x(Q(x) \rightarrow P(x))$$

or

$$Q(x), P(T_1), P(T_2), \dots \dashv\vdash P(x)$$

which is obviously invalid. If it had a finite number of premises, based on the terms T_1, T_2, \dots, T_n , then it could be logically valid with one more premise added, namely

$$\forall x(Q(x) \rightarrow ((x=T_1) \vee \dots \vee (x=T_n)))$$

guaranteeing that T_1, \dots, T_n are all the Q 's there are. However, if the premises and hence the terms are infinite in number, such a premise cannot be formulated and the argument can not even be made valid in this way.

Hence it seems that (ω -1) cannot pretend to *logical* validity, but merely to some more general kind of validity, assuming the intended content of the term *natural number* and all the numerals; and this is also how Etchemendy, 1990, interpreted the original proposal of Tarski (1936). In fact Tarski confessed that he did not see a sharp boundary between logical and extralogical words – and concluded that perhaps the line, and hence the concept of logical consequence is wholly arbitrary. So imagine we draw the boundary so that the arithmetical vocabulary falls on the logical side. Can we say that Tarski's case shows that this ('logico-arithmetic') kind of consequence cannot be captured by inferential rules?

However, even this is implausible. For can we really say that $(\omega-1)$ is an intuitively valid instance of consequence? Well, if intuitive, then only for those who understand what the three dots are supposed to be a shorthand for. But what are they a shorthand for? Shorthands save us labor – allow us to write something short instead of something long. But the three dots of the above inference schema are *indispensable* – we cannot expand them to a full wording. This was observed by Wittgenstein (1953, §208):

Es ist zu unterscheiden: das "u.s.w.", das eine Abkürzung der Schreibweise ist, von demjenigen, welches dies nicht ist. Das "u.s.w. ad inf." ist keine Abkürzung der Schreibweise. Daß wir nicht alle Stellen von π anschreiben können, ist nicht eine menschliche Unzulänglichkeit, wie Mathematiker manchmal glauben¹⁰.

But do we not *understand* the inferential pattern above and do we not *have* the intuition that it is valid? Granted; but how do we manage to understand what is beyond the three dots? We have learned how to write the natural number sequence starting from 0 and going on as far as requested; and analogously we know how to continue the beginning of the list of statements in the antecedent of $(\omega-1)$. But how do we know where to stop? This is the point: we are *never* to stop – and hence there is no way to expand the three dots save with the help of a universal quantification: *The collection of premises consists of 'P(x)' for every natural number x*. Informally expressed, the rule amounts to

$$\frac{\forall x \{ \ulcorner P(x) \urcorner \}}{\forall n P(n)}$$

where, of course, the quasiformula above the line is not an object language statement, but rather a metalinguistic description of the set of premises¹¹.

Hence, one is tempted to say, the instance of consequence which is allegedly not capturable as an instance of inference, *is*, after all, an inference – but one transgressing the boundary of languages, namely that between metalanguage and the object language. It can be seen as the explication of the universal quantification of the object language only if we take for granted that we can make a free use of universal quantification of the metalanguage. In this sense, consequence can be said to be 'a kind of' inference (or perhaps better quasiinference), which, nevertheless, does not fall under what is normally understood as inference proper.

¹⁰ "We should distinguish between the "and so on" which is, and the "and so on" which is not, an abbreviated notation. "And so on ad inf." is not such an abbreviation. The fact that we cannot write down all the digits of π is not a human shortcoming, as mathematicians sometimes think." (Cf. also Peregrin, 1995, Chapter 9.)

¹¹ If metalanguage becomes incorporated into the object language (via gödelization), then such rules can underlie Feferman's (1962) *reflection principles*.

8. Conclusion

It is misguided to see the relationship between consequence and inference as that between the subject matter of logical investigations and their tool; to see consequence as something that exists 'out there' and inference as something that we devise to approximate it. It is, I have argued, inference that should be seen as the basic concept, underlying even the concept of consequence. Therefore, it is more adequate to see consequence as a 'quasiinference', as what becomes of inference if we relax the concept of inferential rule. This also vindicates *inferentialism*: the view that it is inferential patterns that furnish our words with their semantics and that are, consequently, responsible for the entire logical structure of our language.

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