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MEANING AS AN INFERENTIAL ROLE

ABSTRACT. While according to the inferentialists, meaning is always a kind of inferential role, proponents of other approaches to semantics often doubt that actual meanings, as they see them, can be generally reduced to inferential roles. In this paper we propose a formal framework for considering the hypothesis of the "general inferentializability of meaning". We provide very general definitions of both "semantics" and "inference" and study the question which kinds of semantics can be reasonably seen as engendered by inferences. We restrict ourselves to logical constants; and especially to the question of the fesaibility of seeing the meanings of those of classical logic in an inferential way. The answer we reach is positive (although with some provisos).

1. THE INFERENTIALIST TRADITION

Contemporary theoreticians of meaning can be divided, with a degree of oversimplification, into those seeing the meaning of an expression as principally a matter of what the expression denotes or stands for, and those seeing it as a matter of how the expression is used. A prominent place among the latter is assumed by those who seek the basis of meaning in the usage of an expression, in the "language games" we play with it; and a prominent place among *them* is assumed by those who claim that meaning is a matter of the role of the expression w.r.t. the *rules* of the language games, especially the *inferential* rules, which are, as Brandom (1994) points out, crucial for our all-important game of "giving and asking for reasons". From this viewpoint, the meaning of an expression is, principally, its *inferential role*.

Brandom (1985, p. 31) characterizes the inferentialist tradition (which, according to him, can be traced back to Leibniz) in the following way:

> The philosophical tradition can be portrayed as providing two different models for the significances which are proximal objects of explicit understanding, representational and inferential. We may call "representationalism" the semantically reductive view that inference is to be explained away in favor of more primitive representational relations.

> ... By "inferentialism", on the other hand, one would mean the complementary semantically reductive order of explanation which would define representational features of subsentential expressions in terms of the inferential relations of sentences containing them.

Various degrees of commitment to inferentialism can be found also within the writing of some of the founding fathers of analytic philosophy. Thus, Frege's first account for the concept of "conceptual content", which he presents in his *Begriffsschrift* (1879, pp. 2–3), is distinctively inferentialist:

> The contents of two judgments can differ in two ways: first, it may be the case that [all] the consequences which may be derived from the first judgment combined with certain others can always be derived also from the second judgment combined with the same others; secondly this may not be the case ... I call the part of the content which is the *same* in both *the conceptual content*.

Similarly, Wittgenstein assumed a distinctively inferentialist standpoint in a particular stage of the development of his thought from the Tractarian representationalism to the more inclusive use theory of meaning of the *Investigations*. In his *Remarks on the Foundation of Mathematics* (1956, pp. 24, 398) we can read:

> The rules of logical inference cannot be either wrong or right. They determine the meaning of the signs ... We can conceive the rules of inference – I want to say – as giving the signs their meaning, because they are rules for the use of these signs.

Recently, the philosophical foundations of inferentialism have been elaborated especially by Brandom (1994, p. 144):

> It is only insofar as it is appealed to in explaining the circumstances under which judgments and inferences are properly made and the proper consequences of doing so that something associated by the theorist with interpreted states or expressions qualifies as a *semantic* interpretant, or deserves to be called a theoretical concept of a *content*.

Hence, according to Brandom (2000, p. 30), the inferentialist semantic explanations

> beginning with proprieties of inference ... explain propositional content, and in terms of both go on to explain the conceptual content expressed by subsentential expressions such as singular terms and predicates.[1]

All of this indicates that the idea of identifying meanings with inferential roles is worth investigating. However, its viability has been often challenged (see, e.g. Prior 1960/1961, or Fodor and LePore 1993). The most straightforward challenges amount to claiming that the meanings that some of our expressions clearly seem to carry cannot be envisaged as creatures of inference. This invites the general question, which I would like to address in this paper:

(*) *Which kinds of meanings can be conferred on words by means of inferential rules?*

I will neither endorse a specific version of inferentialism, nor argue for inferentialism as a philosophical position, nor discuss its philosophical foundations (I have done so elsewhere – see esp. Peregrin 2001, Chapter VIII, 2004b). Instead, I will focus on establishing a framework which would allow to make the question (*) reasonably precise and I will make some rather technical points which I think are relevant for the debate – though their exact philosophical significance may not be obvious.

I will simply presuppose that languages possess inferential structures, i.e. that in any language worth its name there are some sentences which can be (correctly) inferred from other sentences. (Can we have a language without such a structure? We can have *something*, but I do not think it would be a *language* in a non-metaphoric sense of the word.) I will also presuppose that this structure is not derived from the truth-valuations of the sentences or from truth-conditions. (How, then, does it come into being? Brandom sees it as a kind of commitment- or entitlement-preservation – to say that A is inferable from X is to say that whoever is committed [entitled] to X is committed [entitled] to A. This may lead to a finer inferential apparatus of the kind of that presented by Lance 1995 – but I will not go into these details here.) In addition to inference, we will consider the relation of incompatibility, which sometimes also plays a vital role within inferentialist explanations. (This is connected to another important point related to the very nature of inference. It is often assumed that $X \vdash A$ amounts to a prescription: an obligation to assert A when one has asserted X, to pass over from the thought that X to the thought that A, etc. But this is obviously not the case, for such prescriptions simply could never be obeyed, it being impossible to assert *all* consequences of one's assertion, or think all consequences of one's thought. Hence $X \vdash A$ is much more reasonably construed as a *constraint*: the exclusion of the possibility to deny A when one has asserted X, i.e. the incompatibility of X with the negation of A.)

As it does not seem reasonable to presuppose that for each word there must be a meaning-conferring inferential pattern independent of those of other words, we do not exclude the possibility that the meaning of a word is specifiable only in a mutual dependence with meanings of other words – i.e. that the pattern constitutive of the meaning of a word involves other words. From this viewpoint it might be better to talk, more generally, about furnishing semantics for a language than about conferring meaning on a single word.

(Establishing semantics for a language is conferring meanings on all its words; whereas conferring meaning on one of its words may be inextricable from conferring meanings on other words.)

Hence it may be better if we reformulate the above question as

(**) *Which kinds of semantics are determined by inferential rules?*

However, to be able to deal with this question a rigorous manner, we must first clarify or explicate all the terms occurring in it; and this is the theme for the upcoming sections. Let us start from the term "inferential".

In what follows, I will restrict my attention to logical vocabulary, which offers the most perspicuous stratum of language (especially in the formal languages which we have come to employ to regiment the natural ones). This should not be read as rejecting the possibility of the inferential treatment of other parts of our vocabulary. On the contrary, I believe semantics of any kind of expression can be construed as a kind of "inferential role" (though in the case of empirical words this presupposes extending the concept of inference beyond its usual limits, to "inferences" from the world to language and vice versa[2]). However, here I want to restrict myself to the simplest case of logical words.

2. INFERENCE AND INCOMPATIBILITY

A (*strong*) *inferential structure* is an ordered pair $\langle S, \vdash_S \rangle$, where S is a set whose elements are called *statements* and $\vdash_S$ is a relation between finite sequences of elements of S and elements of S. If the sequence $\langle A_1, \ldots, A_n \rangle$ of statements is in the relation $\vdash_S$ to the statement A then we will write simply

$$A_1, \ldots, A_n \vdash_S A$$

We will use the letters A, A_1, A_2,..., B, C for statements, the letters X, Y, Z for finite sequences thereof, and U, V for sets of statements. If X is a sequence of statements, then X^* will be the set consisting of all its constituent statements.

We define

$$Cn(U) = \{A \mid \text{there is a sequence } X \text{ such that } X^* \subseteq U \text{ and } X \vdash_S A\}$$

We will say that U is *closed* if $Cn(U) = U$.

We will say that $\langle S, \vdash_S \rangle$ is *standard* iff for every X, Y, Z, A, B, C:

(REF) $A \vdash_S A$

(EXT) if $X, Y \vdash_S A$, then $X, B, Y \vdash_S A$

(CON) if $X, A, A, Y \vdash_S B$, then $X, A, Y \vdash_S B$

(PERM) if $X, A, B, Y \vdash_S C$, then $X, B, A, Y \vdash_S C$

(CUT) if $X, A, Y \vdash_S B$ and $Z \vdash_S A$, then $X, Z, Y \vdash_S B$

The properties of $\vdash_S$ spelled out by these schemas will be also called *reflexivity*, *extendability*, *contractibility*, *permutability* and *transitivity*. (The schemas are also known as *identity*, *thinning*, *contraction*, *permutation* and *cut*.)

An *incompatibility structure* is an ordered pair $\langle S, \perp_S \rangle$, where S is a set of statements and $\perp_S$ is a set of finite sequences of elements of S. If the sequence $A_1,...,A_n$ belongs to $\perp_S$, we will write

$$\perp_S A_1, \ldots, A_n$$

We will say that a set U of statements is *consistent* if there is no sequence X such that $X^* \subseteq U$ and $\perp_S X$.

We will say that $\langle S, \perp_S \rangle$ is *standard* iff for every X, Y, Z, A, B, C

(EXT) if $\perp_S X, Y$, then $\perp_S X, A, Y$
(CON) if $\perp_S X, A, A, Y$, then $\perp_S X, A, Y$
(PERM) if $\perp_S X, A, B, Y$, then $\perp_S X, B, A, Y$

Let $\langle S, \vdash_S \rangle$ be an inferential structure. Let us define $\perp_S$ as follows:

$$\perp_S X \equiv_{\text{Def.}} X \vdash_S A \text{ forevery } A.$$

The resulting incompatibility structure $\langle S, \perp_S \rangle$ will be called *induced by* $\langle S, \vdash_S \rangle$.

Let conversely $\langle S, \perp_S \rangle$ be an incompatibility structure. Let

$$X \vdash_S A \equiv_{\text{Def.}} \perp_S Y, X, Z \text{ for every } Y \text{ and } Z \text{ such that } \perp_S Y, A, Z.$$

The resulting inferential structure $\langle S, \vdash_S \rangle$ will be called *induced by* $\langle S, \perp_S \rangle$.

THEOREM 1. If an inferential structure is standard, then the incompatibility structure induced by it is standard. If an incompatibility structure is standard, then the inferential structure induced by it is standard.

PROOF. Most of it is trivial, so let us prove only that if an incompatibility structure is standard, then the induced inference is transitive. Hence we have to prove that if

(i) $\perp_S W, X, A, Y, W'$ for every W and W' such that $\perp_S W, B, W'$,

and

(ii) $\perp_S W, Z, W'$ for every W and W' such that $\perp_S W, A, W'$,

then

(iii) $\perp_S W, X, Z, Y, W'$ for every W and W' such that $\perp_S W, B, W'$.

It is clear that (ii) is equivalent to

(ii′) $\perp_S W, X, Z, Y, W'$ for every W, X, Y and W'such that $\perp_S W, X, A, Y, W'$

and hence to

(ii″) for every X and Y it is the case that $\perp_S W, X, Z, Y, W'$ for every W and W' such that $\perp_S W, X, A, Y, W'$.

And it is clear that (iii) is a consequence of (i) and (ii″). □

A *generalized inferential structure* (*gis*) is an ordered triple $\langle S, \vdash_S, \perp_S \rangle$. It is called *standard* iff the following conditions are fulfilled:

(1) $\langle S, \vdash_S \rangle$ is standard;
(2) $\langle S, \perp_S \rangle$ is standard;
(3) if $\perp_S X$, then $X \vdash_S A$ for every A;
(4) if $X \vdash_S A$ then $\perp_S Y, X, Z$ for every Y and Z such that $\perp_S Y, A, Z$.

A standard gis is called *perfect*, iff it moreover fulfills the following:

(5) if $X \vdash_S A$ for every A, then $\perp_S X$ (i.e. $\perp_S$ is induced by $\vdash_S$)
(6) if $\perp_S Y, X, Z$ for every Y and Z such that $\perp_S Y, A, Z$, then $X \vdash_S A$ (i.e. $\vdash_S$ is induced by$\perp_S$).

Thus, in a perfect structure, incompatibility is reducible to inference (X is incompatible iff everything is inferable from it) and vice versa (A is inferable form X iff everything which is incompatible with A is also incompatible with X).

Let us now prove one more general result concerning standard gis's.

THEOREM 2. Let $\langle S, \vdash_S, \perp_S \rangle$ be a standard gis. Then $Cn(X^*)$ is inconsistent only if $\perp_S X$.

PROOF. Let $Cn(X^*)$ be inconsistent. This means that there exists a sequence $Y = A_1,...,A_n$ of statements such that $Y^* \subseteq Cn(X^*)$ and $\perp_S Y$. This further means that there exist $X_1,...,X_n$ so that $X_i^* \subseteq X^*$ and $X_i \vdash_S A_i$. But due to the extendability and permutability of $\vdash_S$, it follows that $X \vdash_S A_i$. Thus, whatever is incompatible with A_1 must be incompatible with X; hence $\perp_S X, A_2, \ldots, A_n$, and hence, as $\perp_S$ is permutable, it is the case that $\perp_S A_2, \ldots, A_n, X$. Then, as whatever is incompatible with A_2 is incompatible with X, it is the case that $\perp_S A_3, \ldots, A_n, X, X$, and so on. Ultimately, $\perp_S X, \ldots, X$, and, as $\perp_S$ is contractible, it is the case that $\perp_S X$. □

3. INFERENCE AND TRUTH-PRESERVATION

Suppose we have a set V of truth valuations of elements of S. i.e. a subset of $\{0,1\}^S$. (Thus, valuations can be identified with subsets of S.) The pair $\langle S, V \rangle$ will be called a *semantic system*. Then we can define the relation $\vDash_S$ of entailment and the property's of incompatibility as follows:

$X \vDash_S A$ iff $v(A) = 1$ for every $v \in V$ such that $v(B) = 1$ for every $B \in X^*$ $\perp\!\!\!\perp_S X$ iff for no $v \in V$ it is the case that $v(B) = 1$ for every $B \in X^*$

Then $\langle S, \models_S, \perp\!\!\!\perp_S \rangle$ is a gis; and we will say that it is the gis *of* $\langle S, V \rangle$. It is easily checked that this gis is standard.

Let us call a gis $\langle S, \vdash_S, \perp_S \rangle$ *truth-preserving* if there is a V such that $\langle S, \vdash_S, \perp_S \rangle$ is the structure of $\langle S, V \rangle$. We have seen that standardness is a necessary condition of truth-preservingness; now we will show that it is also a sufficient condition – hence that a gis is truth-preserving iff it is standard.

THEOREM 3. A gis is truth-preserving if it is standard.

PROOF. Let V be the class of all closed and consistent subsets of S. We will first prove that then $X \vdash_S A$ iff $X \vDash_S A$. The direct implication is straightforward: if $X \vdash_S A$ and $X^* \subseteq U$ for some $U \in V$, then $A \in Cn\,(U)$ and hence, as U is closed, $A \in U$. So we only have to prove the inverse implication.

Hence let $X \vDash_S A$. This means that whenever $U \in V$ and $X^* \subseteq U$, $A \in U$; i.e. that $A \in U$ for every U such that

(i) $X^* \subseteq U$
(ii) U is consistent (i.e. $Y^* \subseteq U$ for no Y such that $\perp_S Y$)
(iii) U is closed (i.e. $Cn(U) = U$).

As $\vdash_S$ is reflexive, $X^* \subseteq Cn(X^*)$. As it is transitive, $Cn(Cn(X^*)) = Cn(X^*)$. This means that $Cn(X^*)$, in the role of U, satisfies (i) and (iii), and hence if it is consistent, then $A \in Cn(X^*)$. As a consequence we have: either $A \in Cn(X^*)$, or $Y^* \subseteq Cn(X^*)$ for some Y such that $\perp_S Y$. In both cases it must be the case that $Z \vdash_S A$ for some sequence Z all of whose members belong to X^*. Due to the extendability and contractibility of $\vdash_S$, this means that $Y \vdash_S A$ for some sequence Y with the same elements as X and hence, due to the permutability of $\vdash_S$, $X \vdash_S A$.

Now we will prove that $\perp_S X$ iff $\perp\!\!\!\perp_S X$; and as the direct implication is again obvious, it is enough to prove that $\perp\!\!\!\perp_S X$ entails $\perp_S X$. So let it be the case that $\perp\!\!\!\perp_S X$. This means that $X^* \subseteq U$ for no $U \in V$; and as $X^* \subseteq Cn(X^*)$, that $Cn(X^*) \subseteq U$ for no $U \in V$. But as $Cn(X^*)$ is closed and V is the set of all closed and consistent subsets of S, $Cn(X^*)$ is bound to be inconsistent. Hence, according to Theorem 2, $\perp_S X$.

Thereby the proof is finished. □

This means that a structure is truth-preserving if and only if it is standard; and therefore we have a reason to be interested in standard structures: for is not truth-preservation what logic is about? True, for an inferentialist, truth-preservation is not *prior* to inference, but even she would probably want to have inference explicable as truth-preservation – at the end of the day, if not at the beginning. She might want to inverse the order of explanation and claim that 'truth is that which is preserved by inference'. Hence should we pay special attention to *standard* structures?

There may appear to be reasons not to do so. Thus, for example, Lance (1995) and Lance and Kremer (1994, 1996) have forcefully argued that the notion of inference appropriate to the Brandomian inferentialist framework is one that does justice to relevant, rather than classical logic. This indicates that the true inferential structure of natural language might *not* be standard – we should not expect that it will comply to (EXT). However, here I think we must be mindful of an important distinction.

Given a collection R of rules, we can ask what can be inferred (proved, justified, substantiated ...) in their terms. And it seems clear that in the intuitive sense of "inference", A can, for example, always be inferred from A, independently of the nature of R – A is always justified given A, and hence (REF) appears to be vindicated. Also if A

can be inferred – in this sense – from X (A is justified given X), it can be inferred from anything *more* than X (A is justified given any superset X' of X – the justification is the same, simply ignoring any extra elements of X'). Hence from this viewpoint we should also accept (EXT), and similarly all the other structural rules.

This indicates that given a "substandard" inferential relation, the question whether A is inferable from X in terms of the relation is ambiguous. Besides the obvious answer that A is inferable from X iff it is in the given relation to it, there is the response that it is so inferable if it can be obtained from X in terms of the given relation *plus the obvious properties of inference* (in the intuitive sense of the word). Hence we have inference in the narrow, and inference in the wider sense: whereas inferability in terms of R in the narrow sense simply amounts to R, inferability in terms of R in the wide sense amounts to the standard closure of R. Hence special attention for standard inferential structures might be vindicated by the fact that inference in the wider sense *does* inevitably lead to them.

Moreover, it seems to be precisely this closure which interconnects inference with truth-preservation (i.e. *consequence*). We claimed that inference should be construable as truth-preserving not only on the non-inferentialist construal, according to which it is merely an expedient of our account for truth-preservation, but also on the inferentialist account, according to which it *underlies* truth-preservation. This indicates why the basic inferential structure should be "sub-standard": it should be a part of, and extendable to, a standard structure. Furthermore, whereas on the non-inferentialist construal it is simply the case that the closer it is to truth-preservation the better (and it is usually taken for granted that the results of Tarski and Gödel block the possibility of covering the whole of it), on the inferentialist construal this need not be the case. However, there should be a unique way from it to truth-preservation: for the truth-preservation must have arisen from inference.

Hence even if we accept the arguments in favor of the relevantist account for inference, there is a sense in which we can still see inference as standard and thus amounting to truth-preservation. In fact, on the relevantist construal,

$$\textit{inference} = \textit{truth-preservation} + \textit{relevance};$$

but as we do not take truth as more basic than inference, we cannot take the 'underlying' at face value and we must transform this into

$$\textit{truth-preservation} = \textit{inference} - \textit{relevance}$$

And the "subtraction" of relevance seems to amount to forming the standard closure.

Also it seems that what has been just said about inference and inferential structures applies *mutatis mutandis* to incompatibility and incompatibility structures: If we delimit a collection of incompatible sets, then besides a set being incompatible in the narrow sense of being a member of this collection there is again the wider sense in which it is incompatible iff its incompatibility follows from the definition of the collection *plus the obvious properties of incompatibility*. But we should note that *generalized* inferential structures, especially the rules (3) and (4) interconnecting inference and incompatibility, are more problematic from this viewpoint. In fact, they may mark a deeper than just terminological issue between the relevantists and non-relevantists:[3] it seems to be possible that the relevantist, eved if she admitted that there is a sense of "inference" in which inference is always standard (thought this is not the sense she would prefer), and there is a sense on "incompatibility" in which incompatibility is always standard, may still deny that inference and incompatibility in these senses are tied together by the *ex falso quodlibet* rule (3).

There are also different arguments against the standardness of inferential structures. If we take natural language at face value, then we might wonder whether we can take truth-preservation itself as standard. The point is that many sentences of natural language acquire truth-values only when embedded within a context: thus though the statement *He is bald* does not have a truth value by itself, it acquires one when following *The king of France is wise*. Hence we may say that *The king of France is wise* followed by *He is bald* entails *The king of France is bald*, but this statement is surely not entailed by *He is bald* followed by *The king of France is wise*. This gives us a reason to wonder whether we can see truth-preservation itself as complying to (PERM). The standard solution to this, of course, is to restrict logical investigations to those sentences which are *not* context-dependent; but there remains also the possibility of taking it at face value which may lead to an approach to semantics different from the one investigated here (see Peregrin in press-a, for a sketch).

All in all, we conclude that both from the non-inferentialist and from the inferentialist viewpoint we can see inference as "approximating" truth-preservation (though the "approximating" can be taken literally only in the former case). How, generally, can we get truth-preservation out of inference? Well, we saw that, for example, the relation of truth-preservation arising from alleviating the relevance requirement from relevantist inference might amount to the minimal standard relation containing the inferential relation. However, we will take a more general approach to the topic, which will emerge from our explication the of the terms "semantics" and "determines". But beforehand, let us consider one more non-inferentialist aspect of the situation.

4. EXPRESSIVE RESOURCES OF SEMANTIC SYSTEMS

Though from the inferentialist viewpoint, we *establish* semantics by means of inferences, from a more common viewpoint, semantics is something which is here prior to inferences and we use inferences only to "capture" it. From the latter viewpoint, the acceptable truth-valuations of a semantic system delimit what is possible, i.e. represent "possible worlds", and a statement can be semantically characterized by the class of worlds in which it is true. Hence classes of possible worlds are potential semantic values of statements; and languages may differ as to their "expressive power".

Take the system $\langle\{A, B\}, \{\{A, B\}, \{A\}, \{B\}, \emptyset\}\rangle$, i.e. a system with two statements and all possible truth-valuations. If we number the valuations in the order in which they are listed, we can see that the statement A belongs to 1 and 2, whereas B belongs to 1 and 3; hence if we switch to the possible-world-perspective, then A expresses {1,2}, whereas B expresses {1,3}. There is no statement expressing {1}, or {1,4}, or, say, {2,3,4}. This can be improved by extending the language: we can, for example, add a statement C expressing {1}: $\langle\{A, B, C\}, \{\{A, B, C\}, \{A\}, \{B\}, \emptyset\}\rangle$.[4]

To make these considerations more rigorous, we need some more terminology. Let $F = \langle S, V\rangle$ be a semantic system. For every statement A from S, let $|A|$ denote the set of all and only elements of V which contain A; hence let

$$|A| \equiv_{\text{Def.}} \{U \in V \mid A \in U\}$$

A subset V' of V is called *expressible in F* iff there is an $A \in S$ so that $|A| = V'$. F is called (*fully*) *expressible* iff every subset of V is expressible. F is called *Boolean expressible* iff the complement of any expressible set is expressible and the union of any two expressible sets is again expressible. (It is obvious that if a semantic system is Boolean expressible, then its statements can be seen as constituting a Boolean algebra.)

Fully expressible systems constitute a proper subset of the set of semantic systems; however, in some respects we may want to restrict our attention to them. The point is that it seems that if our ultimate target is natural language, then we should not take a lack of expressive resources too seriously. A natural language may, for various contingent reasons, lack some words and consequently some sentences, but this does not seem to be a matter of its 'nature' – natural languages are always flexible enough to take in their stride the creation of new expressive resources whenever needed.[5] Therefore it may often seem reasonable to simply presuppose full expressibility, or at least something close to it (like Boolean expressibility).

Fully expressible systems have some properties which not all semantic systems have. An example is spelled out by the following theorem:

THEOREM 4. The gis of an expressible semantic system is perfect.

PROOF. Let $X \vDash_S A$ for every A. Let # be an element of S such that $|\#| = \emptyset$. Then $X \vDash_S \#$ and hence X cannot be part of any element of V which does not contain #. But as # belongs to no such element, neither can X, and hence $\perp\!\!\!\perp_S X$. Let it now be the case that $\perp\!\!\!\perp_S Y, X, Z$ for every Y and Z such that $\perp\!\!\!\perp Y, A, Z$. Now suppose there is a $U \in V$ such that $X^* \subseteq U$, but $A \notin U$. Let $B \in S$ be such that $|B| = \{U\}$. Then obviously $\perp\!\!\!\perp_S A, B$, but not $\perp\!\!\!\perp_S X, B$. □

We can see that the only properties of the semantic system used in the proof are the expressibility of the empty set and of every singleton. This indicates that we do not need full expressibility; and indeed it can be shown that Boolean expressibility, or even expressibility of the empty set plus the expressibility of the complement of every expressible set is enough.[6]

5. SEMANTIC SYSTEMS AND SEMANTICS

Clearly the question (**) makes sense only provided we have a non-inferentialist explication of "semantics" – i.e. it makes little sense to an inferentialist who denies the possibility of such an explication. However, the fact that we will provide such an independent explication should not be construed as building the rejection of inferentialism into the foundation of our approach. On the contrary, the point of our effort is in checking whether the *prima facie* independence could perhaps be eliminated – and if so, then inferentialism would appear to be *vindicated* (at least to the extent to which we admit that our delimitation of *semantics* is quite general).

So how can we delimit a general concept of semantics independent of inferences? Elsewhere (see Peregrin 1995) I argued that the generalization of the concept of *semantics interpretation*, as the concept is used in logic, comes down to "*minimal*" *compositional mapping verifying some sentences and falsifying others*. I subsequently argued (Peregrin 1997) that different semantic interpretations of this kind can still often be seen as amounting to *the same semantics* (in the intuitive sense of the word); and I concluded that what makes interpretations substantially different are differences within the respective spaces of acceptable truth valuations which they institute. In other words, I came to the conclusion that the most general explication of "semantics" is a space of truth-valuations, i.e. that it is provided by our above concept of *semantic system*. This conclusion accords with the approach pioneered by van Fraassen (1971) and recently elaborated by Dunn and Hardegree (2000). As it would be beyond the scope of the present paper to argue for this at length, I give here only a digest.

Semantic interpretation seems obviously to go hand in hand with a truth-valuation of sentences: sentences (or at least some of them), by being semantically interpreted, become true or false. However, this does not necessarily mean that semantic interpretation fixes the truth values of all sentences – surely a sentence such as "The sun shines" does not become true or false by being made to mean what it does. What semantic interpretation generally does is to impose limits on possible truth-valuations: e.g., it determines that if "The sun shines" is true, then "The sun does not shine" must be false; hence that the sentence "The sun shines and the sun does not shine" is bound to be always false, etc. This means that semantic interpretation should put some *constraints* on the possible truth-valuations of sentences.[7]

Moreover, many philosophers of language (most notably Davidson, 1984) have argued that all there is to meaning must consist in truth conditions. Now let us think about the ways truth conditions can be articulated: we must say something of the form

X is true iff Y,

where X is replaced by the name of a sentence and Y by a description of the conditions – i.e. a sentence. Hence we need a language in which the truth conditions are expressed – a *metalanguage*. However, then our theory will work only so long as we take the semantics of the metalanguage at face value – in fact we will merely have reduced the truth conditions of the considered sentence, X, to a sentence of the metalanguage, namely the one replacing Y. And to require that the semantics of the latter be explicated equally rigorously as that of X would obviously set an infinite regress in motion.

This indicates that it might be desirable to refrain from having recourse to a metalanguage and instead to make do with the resources of the object language, the language under investigation. Hence suppose that we would like to use a sentence of this very language in place of Y. Which sentence should it be? The truth conditions of X are clearly best captured by X itself; but using X in place of Y would clearly result in an uninteresting truism. But, at least in some cases, there is the possibility of using a *different* sentence of the same language. So let us assume that we use a sentence Z in place of Y. Saying "X is true if ..." or "X is true only if ..." with Z in place of the "..." amounts to claiming that X is entailed by Z and that X entails Z, respectively. (Claiming '"Fido is a mammal" is true if "Fido is a dog" is claiming that "Fido is a mammal" is entailed by "Fido is a dog"'.) And claiming that X is entailed by Z in turn amounts to claiming that every truth-valuation which verifies Z verifies also X – or that any truth-valuation not doing so is not acceptable. Hence, the specification of the range of acceptable truth-valuations represents that part of the specification of truth-conditions which can be accounted for without mobilizing the resources of another language.[8]

If we accept this, then the question (**) turns on the relationship between semantic systems (spaces of truth-valuations of sentences) and inferential structures (relations between finite sequence of sentences and sentences), in particular on the way in which the latter are capable of "determining" the former. So let us now turn our attention to this *determining*.

6. THE INFERENTIALIZABILLITY OF SEMANTICS

An inference can be seen as a means of excluding certain truth-valuations of the underlying language: stipulating $X \vdash A$ can be seen as excluding all truth-valuations which contain X and do not contain A. In this sense, every inferential structure determines a certain semantic system (and if we agree that meanings are grounded in truth conditions, thereby it also confers meanings on the elements of the underlying language). And hence the question which kinds of meanings are conferable inferentially is intimately connected with the question which semantic systems can be determined by inferential structures. This leads to the definition: the gis $\langle S, \vdash, \perp\rangle$ *determines* the semantic system $\langle S, V\rangle$, where V is the set of all v which fulfill the following conditions:

(i) if $v(B) = 1$ for every constituent B of X and $X \vdash A$, then $v(A) = 1$;
(ii) if $\perp X$, then $v(A) \neq 1$ for at least one $A \in X^{*}$.

Now the latter question might *prima facie* seem trivial: we have seen that every semantic system has an inferential structure; does this inferential structure not determine this very system? However, the answer is notoriously negative: an inferential structure of a semantic system might determine a *different* semantic system (though, of course, a system which has the same inferential structure).

Let $S = \{A,B\}$ and let V consist of the two "truth-value-swapping" valuations, i.e. the valuations $\{A\}$ and $\{B\}$. Let us consider all the possible instances of inference for S, and for each of them the valuations we exclude by its adoption:

$\vdash A$	$\emptyset, \{B\}$
$\vdash B$	$\emptyset, \{A\}$
$A \vdash A$	–
$B \vdash A$	$\{B\}$
$A, B \vdash A$	–
$B \vdash B$	–
$A \vdash B$	$\{A\}$
$A, B \vdash B$	–

This means that no combination of the inferences is capable of excluding the valuation $\{A, B\}$; and also no combination is capable of excluding $\emptyset$ without excluding either $\{A\}$ or $\{B\}$. In other words, no inferential structure determines the system $\langle\{A, B\},\{\{A\},\{B\}\}\rangle$.

Now consider, in addition, the possible instances of incompatibility, and the valuations excluded by them:

$\perp\emptyset$	$\emptyset, \{A\}, \{B\}, \{A, B\}$
$\perp A$	$\{A\}, \{A, B\}$
$\perp B$	$\{B\}, \{A, B\}$
$\perp A, B$	$\{A, B\}$

With their aid, it becomes possible to exclude $\{A,B\}$, by stipulating $\perp A, B$. However, it is still not possible to exclude $\emptyset$ without excluding either $\{A\}$ or $\{B\}$. Hence no gis determines $\langle\{A, B\},\{\{A\},\{B\}\}\rangle$. Now this appears alarming: for the semantics we have just considered is precisely what is needed to make B into the *negation* of A (and vice versa). This indicates that inferentialism might fall short of conferring such an ordinary meaning as that of the standard negation.

In fact, this should not be surprising at all. What we need to characterize negation is the stipulation that (i) if a statement is *true*, its negation is *false*, and (ii) if a statement is *false*, its negation is *true*. However, what we can stipulate in terms of inferences is that a statement is *true* if some other statements are *true*. In terms of incompatibility we can also stipulate that some statements cannot be true jointly, hence that if some statements are true, a statement is false, which covers (i) – but we still cannot cover (ii).

As can be easily observed, the situation is similar w.r.t. disjunction and implication. In the former case, it is easy to exclude the valuations which make one of the disjuncts true and the disjunction false (by the inferences $A \vdash A \vee B$ and $B \vdash A \vee B$), but we cannot exclude all those which make the disjunction true and both disjuncts false. In the latter, we can easily guarantee that the implication is true if the consequent is true, and that it is true only if the antecedent is false or the consequent is true ($B \vdash A \to B$ and $A, A \to B \vdash B$), but we cannot guarantee that it is true if the antecedent is false.

Does this mean that the standard semantics for the classical propositional calculus is not inferential? And if so, how does it square with the completeness of the very calculus – for does not the completeness proof show that the axiomatic (i.e. inferential) delimitation of the calculus coincides with the semantic one? In fact, it is indeed not inferential, which does not contradict its completeness. The axiomatization of the calculus yields us its inferential structure, but this structure does not determine the semantics of the calculus. As a matter of fact, it determines another semantics, which, however, shares the set of tautologies with the calculus (which is what vindicates the completeness proof).

7. THE GENTZENIAN GENERALIZATION

Let us now adopt a notation different from the one used so far and write

$$X \vdash$$

instead of

$$\perp X.$$

In this way, we can get inference and incompatibility under one roof – starting to treat $\vdash$ as a relation between finite sequences of statements and finite sequences of statements of length not greater than one. The ordered pair $\langle S, \vdash_S \rangle$ with $\vdash_S$ of this kind will be called a *weak inferential structure*. Such a structure will be called *standard* if the following holds (where G is a sequence of statements of length at most one):

(REF) $A \vdash_S A$
(EXT) if $X, Y \vdash_S G$, then $X, B, Y \vdash_S G$
(CON) if $X, A, A, Y \vdash_S G$, then $X, A, Y \vdash_S G$
(PERM) if $X, A, B, Y \vdash_S G$, then $X, B, A, Y \vdash_S G$
(CUT) if $X, A, Y \vdash_S G$ and $Z \vdash_S A$, then $X, Z, Y \vdash_S G$
(EXT′) if $X \vdash_S$, then $X \vdash_S A$

If $\langle S, \vdash_S \rangle$ is a weak inferential structure, then the strong inferential structure which arises out of restricting $\vdash_S$ to instances with non-empty right-hand sides, will be called its *strong restriction*. It is obvious that the strong restriction of a standard weak structure is itself standard. If, on the other hand, we restrict $\vdash_S$ to instances with empty right-hand sides, we get an incompatibility structure, which will be called the *incompatibility restriction* of the original structure. It is easy to show that if a structure is standard, then both its strong restriction and its incompatibility restriction are also standard. Moreover, they make up a standard generalized inferential structure.

The condition (EXT′) indicates that we can add statements on the right-hand side of $\vdash_S$ (of course if we thereby do not make it longer than 1). However, what, then, about relaxing this restriction, i.e. allowing for arbitrary finite sequences on the right side of $\vdash_S$, and letting the right hand side be freely expandable just as the left hand side is? It is clear that what we reach in this way is in fact Gentzen's sequent calculus. The ordered pair $\langle S, \vdash_S \rangle$ with $\vdash_S$ of this kind will be called a *quasiinferential structure*. Such a structure will be called *standard* if the following holds:

(REF) $A \vdash_S A$
(EXT) if $X, Y \vdash_S U, V$, then $X, A, Y \vdash_S U, V$ and $X, Y \vdash_S U, A, V$
(CON) if $X, A, A, Y \vdash_S U$, then, $A, Y \vdash_S U$; if $X \vdash_S U, A, A, V$ then $X \vdash_S U, A, V$
(PERM) if $X, A, B, Y \vdash_S U$, then $X, B, A, Y \vdash_S U$; if $X \vdash_S U, A, B, V$ then $X \vdash_S U, B, A, V$
(CUT) if $X, A, Y \vdash_S U$ and $Z \vdash_S V, A, W$, then $X, Z, Y \vdash_S V, U, W$

If $\langle S, \vdash_S \rangle$ is a quasiinferential structure, then the weak inferential structure which arises out of restricting $\vdash_S$ to instances with right-hand side of length not greater than 1, will be called its *weak restriction*. The strong restriction of this restriction, i.e. the strong inferential structure which arises out of restricting $\vdash_S$ to instances with right-hand side of length precisely 1, will be called its *strong restriction*. Again, it is obvious that if a qis is standard, then both its weak restriction and its strong restriction are standard.The problem with this version of an inferential structure is that there seems to be reason to prefer a single-conclusion inference to the multiple-conclusion one. Thus, if we subscribe to the Brandomian variety of inferentialism, we submit that inferences are originally a matter of treating people as committed or entitled to something, in particular as treating their commitment/entitlement to something as bringing about their commitment/entitlement to something else. And while it is easily imaginable what it takes to treat somebody as implicitly committed/entitled to one or more things (in terms of sanctions and rewards), it is much more complicated to imagine what it would take to treat her as committed/entitled to at least one of many things. Moreover, it seems that the form of our actual arguments is normally based on the single-conclusion notion of inference: as Tennant (1997, 320) puts it, "in normal practice, arguments take one from premises to a single conclusion".[9] Be it as it may, it is quasiinferential structures, in contrast to inferential ones, that are capable of determining any system over a finite set of statements. Hence if we call a semantic system $\langle S, V \rangle$ *finite* iff S is finite, and if we call it *semifinite* iff V is finite, we can claim

THEOREM 5. For every semifinite semantic system $\langle S, V \rangle$ there is a qis $\langle S, \vdash \rangle$ that determines it; moreover, if S is finite, then also $\vdash$ is finite.

PROOF. Let $\langle S, V \rangle$ be a semantic system and let V be finite. Let $V = \{v_1, ... v_m\}$; and let V^* be the set of all valuations of S that do not belong to V. For every $v \in V^*$ we construct what we will call the *critical quasiinference* in the following way. As $v \notin V$, $v \neq v_j$ for each $j = 1, ...,$ m. Hence for each j there is a sentence $A_v^j \in$S such that $v(A_v^j) \neq v_j(A_v^j)$; i.e. such that either (i) $v(A_v^j) = 1$ and $v_j(A_v^j) = 0$, or (ii) $v(A_v^j) = 0$ and $v_j(A_v^j) = 1$. Assume, for the sake of simplicity, that the sentences in case (i) are $A_v^1, \ldots, A_v^k$ and the sentences in case (ii) are $A_v^{k+1}, \ldots, A_v^w$. Let

$$A_v^1, \ldots, A_v^k \vdash A_v^{k+1}, \ldots, A_v^w$$

be the ν-critical quasiinference. It is obviously the case that this quasiinference does not exclude any valuation from V; for given any $\nu_j \in V$, either $j \leq k$, and then $v_j(A_v^j) = 0$, or $j > k$, and then $v_j(A_v^j) = 1$; and in neither case ν_j is excluded. On the other hand, the ν-critical quasiinference excludes ν; for $v(A_v^j) = 1$ when $j \leq k$ and $v(A_v^j) = 0$ when $j > k$. Now it is clear that $\langle S, \vdash \rangle$, where $\vdash$ consists of the ν-critical quasiinferences for every $\nu \in V^*$, determines $\langle S, V \rangle$. It is also clear that if S is finite, then also V^* is finite and hence $\vdash$ consists of only finite number of instances. □

The question which naturally follows now is: what about other systems? Are semantic systems which are not semifinite also determined by qis's? It would seem that many are, but surely not of them are[10] – but we will not go into these questions here. The reason is that determinedness by a qis is not yet what would make a system inferential in the intuitive sense. Hence we will now try to explicate the intuitive concept of inferentiality more adequately.

8. FINITE BASES

Let us now return to the enterprise of explication of the question (**): we have dealt with "semantics", "determined" and "inferential", but we have so far not tackled "rules". The point is that the idea behind inferentialism is that it is *us*, speakers, who furnish expressions, and consequently languages, with their inferential power – we treat the statements as inferable one from another (perhaps by taking one to be committed to the former whenever she is committed to the latter) and as incompatible with each other. The idea is that we have a finite number of rules and that a statement is inferable from a set of other statements if it can be derived from them with the help of the rules.[11]

This means that we should restrict our attention to inferential structures of a specific kind, namely those whose relation of inference is that of *inferability by means of a finite collection of inferential rules*. What is an inferential rule? Let us call an ordered pair whose first constituent is a finite sequence of elements of S and the second an element of S (finite sequence of elements of S) a (*quasi*)*inference over S*. (Hence if $\langle S, \vdash_S \rangle$ is a (quasi)inferential structure, then $\vdash_S$ is a set of (quasi)inferences.) Now if P is a set ("of parameters"), then a (P-) (*quasi*)*inferential rule over S* will be an (quasi)inference over S in which some elements of S are replaced by elements of P.[12] (We will usually write $X \vdash Y$ instead of the less perspicuous $\langle X, Y \rangle$). An *instance of a(quasi)inferential rule over S* will be any (quasi)inference over S which can be gained from the rule by a systematic replacement of the elements of P by the elements of S.

(REF), for example, is an inferential rule:

$$A \vdash A.$$

Its quasiinferential form, then, is an example of a quasiinferential rule:

$$X \vdash X.$$

However, more interesting rules emerge only when we assume that the set of statements is somehow structured. If, for example, for every two statements A and B there is a statement denoted as $A \wedge B$, we can have the pattern

$$A \wedge B \vdash A$$

$$A \wedge B \vdash B$$

$$A, B \vdash A \wedge B$$

establishing $A \wedge B$ as the conjunction of A and B.

So let us assume we have fixed some inferential rules, and the relation of inference which interests us is the one which 'derives from' them. How? We obviously need some way of inferring inferences from inferences, some *metainferences* or *metainferential rules*. Hence we introduce the concept of *meta(quasi)inference over S*, which is an ordered pair whose first constituent is a finite sequence of inferences over S and whose second is an inference over S. A *(P-)meta(quasi)inferential rule over S* will be a meta(quasi)inference over S with some elements of S in its constituents replaced by those of P. We will separate the antecedent from the consequent of such a rule by a slash and we will separate the elements of its antecedent by semicolons. Thus, the metainferential rule constituted by (CUT) will be written down as follows:

$$X, A, Y \vdash U; Z \vdash V, A, W / X, Z, Y \vdash V, U, W$$

Now what we want is that the inference relation derives from the basic finitely specified inferential rules by means of some finitely specified metainferential rules: A *(quasi)inferential basis* is an ordered triple $\langle S, R, M \rangle$, where S is a set, R is a finite set of (quasi)inferential rules over S and M is a finite set of meta(quasi)inferential rules over S. (Let us assume that all metarules in M have a non-empty antecedent – for metarules with the empty antecedent can be treated simply as rules and put into R.) The *(quasi)inferential structure generated by* $\langle S, R, M \rangle$ is the (quasi)inferential structure whose (quasi)inferential relation is the smallest class of (quasi)inferences over S which contains all instances of elements of R and is closed to all instances of elements of M. A (quasi)inferential structure is called *finitely generated* iff it is generated by a (quasi)inferential basis. A semantic system is called *finitely (quasi)inferential* iff it is determined by a finitely generated (quasi)inferential structure.

Now it is clear that as far as *finite* languages are concerned, (quasi)inferentiality and finite (quasi)inferentiality simply coincide.

THEOREM 6. Every finite (quasi)inferential semantic system is finitely (quasi)inferential.

PROOF. If the number of statements is finite, then there obviously is only a finite number of (quasi) inferences non-equivalent from the viewpoint of the determination of the system. □

The situation is, of course, different in respect to infinite languages. Take the semantic system constituted by the language of Peano arithmetic (PA) and the single truth-valuation which maps a statement on truth iff it is true in the standard model. This system is (trivially) inferential: the needed inferential relation consists of all inferences which have the empty antecedent and a statement of PA true in the standard model in the consequent. However, as the class of statements true within the standard model is not recursively enumerable, the semantic system is surely not *finitely* inferential.

But in fact it seems that inferentiality in the intuitive sense amounts to more than delimitation by *any* kind of a finitely inferential system. The inference relation of the systems we aim at should be derived from the basic inferential rules not by just *any* metainferential rules, but in a quite specific way. If R is a set of inferential rules, then we want to say that A is inferable, by means of R, from X iff there is a sequence of statements ending with A and such that each of its element is either an element of X or is the consequent of an instance of a rule from R such that all elements of the antecedent occur earlier in the sequence. (REF is, strictly speaking, an inferential, rather than a *meta*inferential rule. But we can regard it as a metainferential rule with an empty antecedent.) This amounts to M consisting of the five Gentzenian structural rules. Indeed, A is inferable from X by means of R if and only if the inference $X \vdash A$ is derivable from R by these rules. □

THEOREM 7. A is inferable from X by means of the rules from R (in the sense that there exists a 'proof') just in case X is inferable from R by means of REF, CON, EXT, PERM and CUT.

PROOF. As the proof of the inverse implication is straightforward, let us prove only the direct one. Hence let A be inferable from X. This means that there is a sequence $A_1,\ldots,A_n$ of statements such that $A_n = A$ and every A_i is either an element of X or is inferable by a rule from R from statements which are among $A_1,\ldots,A_{i\text{-}1}$. If $n = 1$, then there are two possibilities: either $A \in X^*$ and then $X \vdash A$ follows from REF by EXT; or A is a consequent of a rule from R with a void antecedent, and then $\vdash A$ and hence $X \vdash A$ due to EXT. If $n > 1$ and A_n is inferable from some $A_{i_1},\ldots,A_{i_m}$ by a rule from R, then $A_{i_1}, \ldots, A_{i_m} \vdash A$, where $X \vdash A_ij$ for $j = 1,\ldots,m$. Then $X, \ldots, X \vdash A$ due to CUT, and hence $X \vdash A$ due to PERM and CON. □

This leads us to the following definition: We will call a (quasi)inferential basis *standard* iff R contains REF and M contains CON, EXT, PERM and CUT (hence if the (quasi)inferential basis is standard, then the (quasi)inferential structure which it generates is standard in the sense of the earlier definition). And we will call it *strictly standard* iff, moreover, M contains no other rules. A (quasi)inferential structure will be called strictly standard if it is generated by a strictly standard (quasi)inferential basis. (Hence every standard, and especially every strictly standard inferential structure is finitely generated.)

It seems that in stipulating inferences we implicitly stipulate also all the inferences which are derivable from them by the structural rules – hence we should be interested only in structures which are strictly standard, or at least standard. It might seem that it is strictly standard inferential systems which are *the* most natural candidate for the role of the explicatum of an "inferential semantic system"; however, the trouble is that no finitely inferential system (and hence surely no standardly inferential one) is capable of accommodating the simplest operators of classical logic.

Though it is possible to fix the usual truth-functional meaning of the classical conjunction by means of the obvious inferential pattern, the same is not possible, for reasons sketched earlier (§6), for the classical negation and nor for the classical disjunction and implication. What *is* possible is to fix the truth-functional meanings of all the classical operators by means of *quasi*inferential patterns, e.g. in this way:

$$A, \neg A \vdash$$
$$\vdash A, \neg A$$
$$A \vdash A \vee B$$
$$B \vdash A \vee B$$
$$A \vee B \vdash A, B$$
$$B \vdash A \to B$$
$$A, A \to B \vdash B$$
$$\vdash A, A \to B$$

Hence (as discovered already by Gentzen) all of classical logic is strictly standardly quasiinferential. Nevertheless, it is *not* strictly standardly inferential.

For an inferentialist, this situation need not be too embarrassing even if she wants to restrict herself to single-conclusion inferences. I have independent reasons, she might claim, to believe that the only (primordial) way to furnish an expression with a meaning is to let it be governed by inferential rules; so if there are 'meanings' which are not conferable in this way, they are not meanings worth the name. But things are not this simple. We have seen that many meanings of a very familiar and seemingly indispensable kind fall into the non-inferential category. Classical negation or disjunction; not to mention the standard semantics for arithmetic. Is the inferentialist saying that these are non-meanings?

To be sure, the inferentialist *may* defend the line that the only 'natural' meanings are the straightforwardly inferential ones; and that all the others are late-coming products of our artificial language-engineering. She might claim that the only 'natural' logical constants are some which are delimitable inferentially (presumably the intuitionist ones), and that the classical ones are their artificial adjustments available only after metalogical reflections and through explicit tampering with the natural meanings.

However, if she does not want to let classical logic go by the board in this way, she appears to have no choice but to settle for (strictly standardly) quasiinferential systems. The latter, as we saw, are strong enough for the classical operators, but as pointed out above, there are reasons to see the multiple-conclusion sequents as less natural than the single-conclusion ones. Fortunately there is a sense in which every strictly standardly quasiinferential system can be regarded as a (standardly) inferential system, so we may after all enjoy some advantages of the quasiinferential systems without officially admitting the multi-conclusion sequents.

9. THE EMULATION THEOREM

What exactly we will show now is that for every strictly standardly quasiinferential system there exists a standardly inferential system with the same class of inferences (and hence especially tautologies). First, however, we need some more definitions: If $X \vdash A_1, \ldots, A_n$ is a quasiinferential rule over S, then its *emulation* will be the metainferential rule $YA_1Z \vdash B; \ldots; YA_nZ \vdash B / YXZ \vdash B$ (if $n=0$, i.e. if the rule is of the shape $X \vdash$, then the antecedent of the emulation is empty). An *emulation* of a quasiinferential basis $\langle S, R, M\rangle$ will be the inferential basis $\langle S, R', M'\rangle$ such that R' is the set of all those elements of R which are inferential (i.e. not quasiinferential) rules, and M' is the union of the set of restrictions of all elements of M to inferences proper and the emulations of all elements of R which are proper quasiinferential rules.

Now we are going to prove that an emulation of a strictly standard quasiinferential basis $\langle S, R, M\rangle$ generates an inferential structure which is identical to the structure which results from taking the qis generated by $\langle S, R, M\rangle$ and dropping all genuine quasiinferences:

THEOREM 8. The emulation of a strictly standard quasiinferential structure is its strong restriction.

PROOF. Let $\langle S, \vdash_S\rangle$ be the quasiinferential structure generated by $\langle S, R, M\rangle$ and let $\langle S, \vdash_S^*\rangle$ be the inferential structure generated by $\langle S, R', M'\rangle$. What we must show is that for every sequence X of elements of S and every element A of S it is the case that $X \vdash_S A$ iff $X \vdash_S^* A$. Let us consider the inverse implication first. As R' is a subset of R, it is enough to show that every metainferential rule from M' which is not an element of M preserves $\vdash_S$, i.e. that for every such rule $X_1 \vdash A_1; \ldots; X_n \vdash A_n / X \vdash A$ it is the case that if $X_1 \vdash_S A_1, \ldots, X_n \vdash_S A_n$, then also $X \vdash_S A$. However, each metainferential rule which is an element of M' but not of M must be, due to the definition of the former, an emulation of a quasiinferential rule from R, i.e. must be of the form $YA_1Z \vdash B; \ldots; YA_nZ \vdash B \,/\, YXZ \vdash B$, where

$X \vdash A_1, \ldots, A_n$ belongs to R. Hence what we have to prove is that if $X \vdash A_1, \ldots, A_n$ belongs to R (and hence $X \vdash_S A_1, \ldots, A_n$) and $YA_1Z \vdash_S B, \ldots, YA_nZ \vdash_S B$, then $YXZ \vdash_S B$. But this easily follows from the standardness of $\vdash_S$ – if $n = 0$, then simply by (EXT), and if $n > 0$, then in the following way:

$X \vdash_S A_1, \ldots, A_n$	assumption
$YA_1Z \vdash_S B$	assumption
$YXZ \vdash_S BA_2, \ldots, A_n$	(CUT)
$\ldots$	
$Y \ldots YXZ \ldots Z \vdash_S B \ldots B$	(CUT)
$YXZ \vdash_S B$	(PERM) and (CON)

Hence if $X \vdash_S^* A$, then $X \vdash_S A$.

The proof of the direct implication, i.e. of the fact that if $X \vdash_S A$, then $X \vdash_S^* A$, is trickier. We will prove that if $X \vdash_S A_1, \ldots, A_n$ and $YA_1Z \vdash_S^* B, \ldots, YA_nZ \vdash^* B$, then $YXZ \vdash_S^* B$. From this it follows that if $X \vdash_S A$ and $YAZ \vdash_S^* B$, then $YXZ \vdash_S^* B$; and in particular that if $X \vdash_S A$ and $A \vdash_S^* A$, then $X \vdash_S^* A$; and as $A \vdash_S^* A$ due to (REF), $X \vdash_S A$ entails $X \vdash_S^* A$.

First, we will need some notational conventions. If $X = A_1,\ldots,A_n$ then we will use

$$X \vdash\vdash_S^* Y$$

as a shorthand for

$$A_1 \vdash_S^* Y; \ldots; A_n \vdash_S^* Y.$$

Moreover,

$$[U]X[V] \vdash\vdash_S^* Y$$

will be the shorthand for

$$UA_1V \vdash_S^* Y; \ldots; UA_nV \vdash_S^* Y$$

Hence now what we need to prove is that for every X and $A_1,\ldots, A_n$ such that $X \vdash_S A_1, \ldots, A_n$, it is the case that $YA_1Z \vdash_S^* B, \ldots, YA_nZ \vdash^* B$ entail $YXZ \vdash_S^* B$ for every Y, Z and B. We will proceed by induction. First, assume that $X \vdash A_1, \ldots, A_n$ belongs to R. Then if $n \neq 1$, then M' contains its emulation, i.e. the metainferential rule $YA_1Z \vdash B, \ldots, YA_nZ \vdash B / YXZ \vdash B$. This means that if $YA_1Z \vdash_S^* B, \ldots, YA_nZ \vdash_S^* B$, then indeed $YXZ \vdash_S^* B$. If, on the other hand,

$n=1$, then $X \vdash A$ belongs to R' and hence $X \vdash_S^* A$; and the fact that $YAZ \vdash^* B$ entails $YXZ \vdash^* B$ follows by (CUT).

Now assume that $X \vdash A_1, \ldots, A_n$ is the result of an application of a metaquasiinferential rule from M. As $\langle S, R, M \rangle$ is strictly standard, the only possibilities are CON, EXT, PERM and CUT. We will prove only the less perspicuous case of CUT. Hence assume that $X \vdash A_1, \ldots, A_n$ can be written in the form $X, Z, Y \vdash_S V, U, W$ so that

$$X, A, Y \vdash_S U, \quad \text{and} \quad \vdash_S V, A, W;$$

and, by induction hypothesis, that

1. $[M]U[N] \vdash\vdash_S^* B$ entails$M, X, A, Y, N \vdash_S^* B$, and
2. $[M]V, A, W[N] \vdash\vdash_S^* B$ entails $M, Z, N \vdash_S^* B$.

What we want to prove is that then $[M]V, U, W[N] \vdash\vdash_S^* B$ entails $M, X, Z, Y, N \vdash_S^* B$. Hence assume that $[M]V, U, W[N] \vdash\vdash_S^* B$. This is to say, we assume

3. $[M]V[N] \vdash\vdash_S^* B$,
4. $[M]U[N] \vdash\vdash_S^* B$, and
5. $[M]W[N] \vdash\vdash_S^* B$.

3. and 5. yield, via (EXT),

6. $[M, X]V[Y, N] \vdash\vdash_S^* B$, and
7. $[M, X]W[Y, N] \vdash\vdash_S^* B$;

whereas 4. and 1. yield

8. $M, X, A, Y, N \vdash_S^* B$.

Now 6, 7 and 8 together amount to

9. $[M, X]V, A, W, [Y, N] \vdash\vdash_S^* B$,

from which we get

$$M, X, Z, Y, N \vdash^* B$$

via 2. □

COROLLARY. For every strictly standardly quasiinferential system there exists a standardly inferential system with the same class of inferences. Hence especially for every such system there exists a standardly inferential system with the same class of tautologies.

10. AN EXAMPLE: CLASSICAL PROPOSITIONAL LOGIC IN THE LIGHT OF INFERENCES

We saw that not even the classical propositional logic (CPL) is inferential in the sense that there are inferences which can delimit the very class of truth-valuations that is constituted by the usual explicit semantic definition of CPL. However, the corollary we have just proved tells us that there is a standard inferential structure which, while not determining the semantics of CPL, *does* determine a semantic system possessing the same class of tautologies. Which inferential structure is it?

It is easy to see that if we base CPL on the primitive operators $\neg$ and $\wedge$, the semantics of CPL is determined by the following quasiinferential rules:

(1) $A \wedge B \vdash A$
(2) $A \wedge B \vdash B$
(3) $A, B \vdash A \wedge B$
(4) $A, \neg A \vdash$
(5) $\vdash A, \neg A$

What we must do is to replace the genuine quasiinferential rules (i.e. those not having exactly one single statement in the consequent, (4) and (5)) with their emulations. This is to say that we must replace (4) and (5) by

(4′) $A, \neg A \vdash B$
(5′) $X, A \vdash B; X, \neg A \vdash B / X \vdash B$

Note that in view of the fact that (1), (2), (3), (4′) and (5′) constitute a possible axiomatization of CPL, the fact that they determine the tautologies of CPL amounts to the completeness result for the logic. But it is, in a sense, more general than the usual one and it throws some new light on the fact that the axioms of classical logic, despite their completeness, do *not* pin down the denotations of the operators to the standard truth-functions. (The point is that the axioms are compatible even with some non-standard interpretations – with negations of some falsities being false and with disjunctions of some pairs of falsities being true. What *is* the case is that if the axioms hold *and if the denotations of the operators are truth functions*, then they are bound to be the *standard* truth functions. But the axioms are compatible with the indicated *non-truth-functional* interpretation of the constants.[13]) From our vantage point we can see that classical logic is complete in the sense that its axioms determine a semantics with the class of tautologies which is the same as that of the standard semantics of CPL; that they, however, do *not* determine this very semantics.

Let us give some illustrations of how proofs within (1)–(5) get emulated by those within (1)-(3), (4′) and (5′). Consider the inference

$$\neg\neg A \vdash A,$$

which is valid in CPL. With (4) and (5) it can be proved rather easily:

1. $\neg A, \neg\neg A \vdash$	(4)
2. $\vdash A, \neg A$	(5)
3. $\neg\neg A \vdash A$	from 1 and 2 by (CUT)

This gets emulated as follows:

1. $\neg\neg A, \neg A \vdash A$	from(4′) by(PERM)
2. $\neg\neg A, A \vdash A$	from (REF) by (EXT)
3. $\neg\neg A \vdash A$	from 1, 2 by (5′)

Or consider the proof of the theorem
$\neg(A \wedge \neg A)$

1. $A \wedge \neg A \vdash A$	(1)
2. $A \wedge \neg A \vdash \neg A$	(2)
3. $A, \neg A \vdash$	(4)
4. $A \wedge \neg A \vdash$	from 1, 2 and 3 by (CUT) and(CON)
5. $\vdash A \wedge \neg A, \neg(A \wedge \neg A)$	(5)
6. $\vdash \neg(A \wedge \neg A)$	from 4 and 5 by (CUT)

The emulation now looks as follows:

1. $A \wedge \neg A \vdash A$	(1)
2. $A \wedge \neg A \vdash \neg A$	(2)
3. $A, \neg A \vdash \neg(A \wedge \neg A)$	(4′)
4. $A \wedge \neg A \vdash \neg(A \wedge \neg A)$	from 1, 2 and 3 by (CUT) and (CON)
5. $\neg(A \wedge \neg A) \vdash \neg(A \wedge \neg A)$	(REF)
6. $\vdash \neg(A \wedge \neg A)$	from 4 and 5 by (5′)

This means that classical logic may be seen as inferential, though in a rather weak sense: there is no inferential way of delimiting its very space of acceptable truth-valuations; however, there *is* a way of delimiting a space of truth valuations which is equivalent to it w.r.t. tautologies (and more generally single-conclusion inferences).

11. EXTREMALITY CONDITIONS

Hence what seems to be a good candidate for the explication of the intuitive concept of "inferential semantics" is the concept of standardly inferential semantic system, i.e. a system generated by a collection of inferential and metainferential rules containing the Gentzenian structural rules. This is obviously of a piece with the ideas of the natural deduction program (Prawitz 1965; Tennant 1997; etc.) and so it would seem that the inferentialist agenda should display a large overlap with the agenda of this program. We have also seen that there is a direct way from the natural quasiinferential characterization of structural operators to their superstandardly inferential characterization.

Let us consider disjunction. $A \vee B$ is partly characterized by the inferential rule

$$A \vdash A \vee B$$
$$B \vdash A \vee B$$

but the characterization has to be completed by the genuine *quasi*-inferential rule

$$A \vee B \vdash A, B$$

This rule gets emulated as

$$A \vdash C; B \vdash C / A \vee B \vdash C$$

which yields us the metainferential characterization of disjunction well-known from the systems of natural deduction. Note that the metainferential rule can be looked at as a "maximality condition". Let us say that the statements A, B, C fulfill the condition $\Phi(A, B, C)$ iff $A \vdash C$ and $B \vdash C$. Then $A \vee B$ can be characterized in terms of the following two conditions:[14]

(i) $\Phi(A, B, A \vee B)$; and
(ii) $A \vee B$ is the strongest statement such that $\Phi(A, B, C)$; i.e. $A \vee B \vdash C$ for every C such that $\Phi(A, B, C)$

Why is this interesting? Because this kind of maximality condition could perhaps be seen as implicit to the statement of an inferential pattern. When we state that $A \vdash A \vee B$ and $B \vdash A \vee B$ and when we, moreover, put this forward as an (exhaustive) characterization of $A \vee B$, we insinuate $\Phi(A, B, ...)$ is fulfilled by $A \vee B$ *and nothing else*.

However, it is clear that if $A \vee B$ fulfills $\Phi(A, B, \ldots)$, then everything weaker (i.e. entailed by $A \vee B$) does, so the "nothing else" can only mean "nothing non-weaker" – hence it means that for every such statement C it must be the case that $A \vee B \vdash C$.

Imagine I am asked what children I have – i.e. to characterize the class of my children – and I answer "I have a son and a daughter". Strictly speaking I am *not* giving a unique characterization of the class – I am only stating that this class contains a boy and a girl. But as it is normally expected that what I say should yield an exhaustive characterization, my statement would be taken to imply (by way of what Grice called a *conversational implicature*) that the class in question is the *maximal* one fulfilling the condition I state. And a similar maximality implicature can be seen as insinuated by my stating that Φ is the pattern characteristic of disjunction.

More to the point, this train of thought appears also to motivate Gentzen's insistence that it is only *introductory* rules which semantically characterize the operators. As Koslow (1992, Section 2.1) shows, it is natural to see it precisely in terms of extremality conditions: the introduction rule yields the elimination rule via the assumption that the introduction rule gives all that there is to the 'inferential behavior' of the connective. Hence it seems that we *can*, after all, delimit the classical disjunction by an inferential pattern – if we assume the maximality implicature. This indicates that instead of allowing for the non-structural metainferential rules (which amounts to passing over from strictly standard to merely standard inferential structures) we could perhaps admit that stating an inferential pattern involves stating the maximality of the operator fixed by the pattern.

Can we see all the other classical logical operators analogously? Gentzen (1934) himself gives the example of implication: $A \rightarrow B$ is the maximal statement which holds whenever B is derivable from A. So here $\Phi(A, B, C)$ would be

$$\begin{array}{c} A \\ \vdots \\ \underline{B} \\ C \end{array}$$

Alternatively, we can characterize implication as the *minimal* operator fulfilling

$$A, A \rightarrow B \vdash B$$

i.e. as an operator fulfilling this and, moreover

$$A, C \vdash B / C \vdash A \rightarrow B$$

But, does not the replacement of maximality by minimality spoil the intuitive picture outlined above, where maximality reflected exhaustiveness? Not really; for as $A \rightarrow B$ is now within the *antecedent* of the basic pattern, exhaustiveness comes to yield not maximality, but minimality. The basic pattern now reads:

$A \rightarrow B$ is such that together with A it entails B.

And assuming exhaustivity,

$A \rightarrow B$, *and nothing else*, is such that together with A it entails B.

Now it is clear that if something follows from $A \rightarrow B$, then it follows from anything stronger, so the "nothing else" makes sense only as "nothing non-stronger". Hence the exhaustivity boils down to

if C does the same, then it is stronger, i.e. $C \vdash A \rightarrow B$.

We can also characterize conjunction as the maximal operator fulfilling

$$C \vdash A$$
$$C \vdash B.$$

Negation, if we want it to be classical, is unfortunately more fishy. It seems that the only pattern available is

$$A, C \vdash B$$
$$\neg C \vdash A.$$

which itself contains the negation sign to be determined; and this appears to largely spoil the picture. Is there a remedy?

We could, perhaps, trade the second part of the negation-pattern, i.e. the law of double negation, for something else – e.g. for the "external" assumption that all our operators are truth-functional. It is clear that the only truth-function which always maps a statement on its maximal incompatible is the standard negation (see Peregrin 2003 for more details). But a more frank solution would be to simply strike out the law of double negation without a substitute. What would be the result? Of course the *intuitionist* negation and consequently the *intuitionist* logic. This indicates, as I have discussed in detail elsewhere (see Peregrin 2004a), that it is *intuitionist* logic that is *the* logic of inference. In this sense, classical logic is not natural from the inferentialist viewpoint (however, its unnaturalness from this viewpoint is outweighed – and maybe overridingly so – by its simplicity).

12. CONCLUSION

Arguing for *inferentialism*, we must first specify what exactly we mean by the term: there are several options. In this paper I have tried to indicate that two of the options can be merged into a single one, which, in its turn, is the hottest candidate for becoming *the* inferentialism. The winner is the "superstandard inferentialism", capable of 'emulating' and hence treatable as encompassing "standard quasiinferentialism". On the technical side, it comes down to the framework of natural deduction. (Its immediate stricter neighbor, "standard inferentialism" is obviously much too weak; while the stronger "quasiinferentialism" appears to be less natural.)

If we accept this, then we should also see intuitionist logic as *the* most natural logic. However, as we have taken pains to indicate, this does not preclude the way to classical logic, which is surely natural in some other respects and whose utter inaccessibility would be, I believe, a failure of inferentialism. Inferentialism is a *descriptive* project concerned with the question *what is meaning?*; whereas the natural deduction program is more a *prescriptive* program concerned with the question *How should we do logic?*. Thus while the latter could perhaps simply ban classical logic if it concluded that one can make do without it, the former is bound to take the extant meanings at face value and face the question *If meaning is an inferential matter, then how could there be meanings that are* prima facie *"non-inferential"*? Hence I think that inferentialism, though it may "favor" some meanings over others, does not result in any unnatural "semantic asceticism". I am convinced that the thesis that all meanings are – more or less direct – creatures of inferences is viable.

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NOTES

[1] See also Lance (1996, 2001) and Kalderon (2001).

[2] See Peregrin (in press-b) for more details.

[3] As was pointed out to me by Michael Kremer.

[4] To be precise, we now have different truth-valuations, since now we are evaluating three instead of two statements. So "expressing {1}" should be read as "expressing a truth-valuation which yields {1} when restricted to $\{A,B\}$".

[5] Consider the recurring discussions about substitutional vs. objectual quantification. The basic problem would seem to be that we simply cannot assume that all entities (including those not known to anybody) within our universe must have names. But this is a red herring (independently of whichever side of the quarrel we stand), for what the proponent of substitutional quantification needs to assume is not that every entity is *named*, but that it is *nameable* – in the sense that language has the resources to form a name as soon as it becomes needed (cf. Lavine 2000).

[6] Moreover, it can be shown that each standard gis such that there is a function f mapping sentences on sentences so that for every X, A and B, $\perp A$, $f(A)$ and if $X,A \vdash B$ and $X, f(A) \vdash B$, then $X \vdash B$, is perfect. (This is important for f is the usual proof-theoretic notion of negation.) I owe this observation to Michael Kremer.

[7] Obviously in the case of such *extensional* languages as those of the predicate calculus, the constraints exclude all valuations save a single one. However, this is clearly not be the case for any natural language.

[8] Of course when dealing with *empirical* terms and *empirical* languages, then we need a way to "connect them with the world" – hence we need either a trusted metalanguage capable of mediating the connection, or else a direct connection which, however, can be established only practically.

[9] This is not supposed to be a knock-down argument against the multiple-conclusion inference (see, e.g., Restall in press, for a defense). However, naturalness clearly is on the side of the single-conclusion one.

[10] A simple example was suggested to me by the referee of this paper: Suppose V consists of all valuations which make only a finite number of sentences of the infinite set S true. Then there is obviously no quasiinference which would exclude an unacceptable valuation without excluding also an acceptable one.

[11] Note that this does *not* mean that we have to be aware of all the rules binding us: the rules we adopt engender other rules and we do not have to foresee all the consequences.

[12] Note that we do not *require* that any particular sentences are replaced by parameters – hence it is even possible for a rule to contain no parameters whatsoever.

Hence we *do not* require that rules be purely "formal"; and the fact that in what follows we will deal especially with rules of this kind should be seen as a matter of the fact that we will restrict our attention to the semantics of logical constants. The rules of inference underlying the semantics of other expressions will surely be non-formal.

[13] This is a fact noted already by Carnap (1943) but rarely reflected upon – see Koslow (1992, Chapter 19), for a discussion.

[14] This form is borrowed from Koslow (1992), whose book offers a thorough discussion of the technical side of the issues hinted at in this section.

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